African Pythagoras:
A Study in Culture and Mathematics Education

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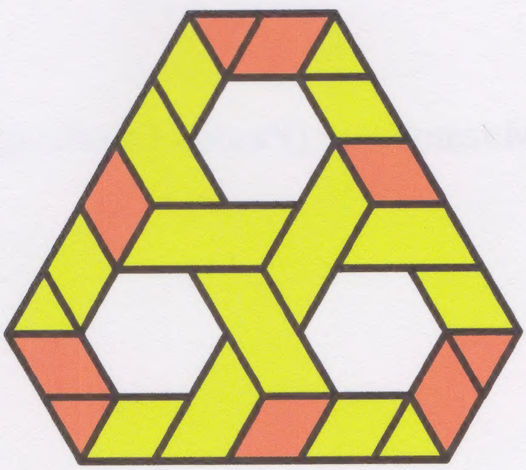

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Culture, Mathematics, Education

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2011

Title: **AFRICAN PYTHAGORAS:**

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The first edition of ***African Pythagoras*** was published in 1992 in Portuguese and in 1994 in English. Some parts of the book were reproduced in Chapter 2 'From African designs to discovering the Pythagorean Theorem' (pp. 54-86) of the author's book ***Geometry from Africa: Mathematical and Educational Explorations*** (Mathematical Association of America, Washington DC, 1999) ('Outstanding Academic Book,' Choice Magazine, 2000).

The 2011 edition of ***African Pythagoras*** is the first colour edition and contains an afterword by Jens Hoyrup (Roskilde University, Denmark).

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PREFACE

The so-called 'Pythagorean Theorem', which asserts that "in any right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the two sides", is "one of the most attractive, and certainly one of the most famous and most useful, theorems of elementary geometry" (Eves, 1983, p. 26). Although legend has ascribed the theorem to Pythagoras of Samos (sixth century B.C.), ancient Babylonians were aware of it over a thousand years before Pythagoras' time. Knowledge of the proposition also appears in ancient Hindu and Chinese works that may go back to the time of Pythagoras, if not earlier.

The title ***AFRICAN PYTHAGORAS*** may intrigue the reader (cf. Swetz' and Kao's provocative title *Was Pythagoras Chinese?* 1977). The historical figure Pythagoras was a Greek and not an African. However, he probably learnt the theorem during his long stay in Egypt (cf. Diop, 1981, p. 436, 479). Nevertheless, the main objective of this study is not the historical analysis of the discovery and the spread of the so-called 'Pythagorean Proposition.' Historical hypotheses are included in the study, but its main objective is ***cultural-didactic***.

African countries see themselves faced with relatively low 'levels of attainment' in mathematics. One of the reasons lies in the fact that many pupils experience mathematics – e.g. the Pythagorean Theorem – as something alien, something rather useless, something difficult and boring, coming from outside Africa. In order to surpass the cultural-psychological learning blockage, it is necessary to revise the curriculum. The objectives, contents and methods of mathematics teaching have to be embedded into the cultural environment of the pupils. The incorporation of ethnomathematics – all types of mathematical activities and reasoning found in daily life – into the curriculum contributes towards this end. In addition, diverse cultural elements may be used as *starting points* for playing and doing interesting mathematics in and around the classroom. The valuing of

the home culture of the child makes him/her feel more confident in his/her capacities. Somebody with confidence in him/herself and his/her culture will learn more easily.

The aim of ***AFRICAN PYTHAGORAS*** is to show how diverse African ornaments and artefacts may be used to create an attractive context for the discovery and the demonstration of the Pythagorean Theorem and of related ideas and propositions.

The author hopes that ***AFRICAN PYTHAGORAS*** may stimulate mathematics teachers, future mathematics teachers and didacticians, to search for further ways to *Africanize* mathematics teaching.

Chapter 1

DID ANCIENT EGYPTIAN ARTISANS KNOW HOW TO FIND A SQUARE EQUAL IN AREA TO TWO GIVEN SQUARES? *

Of all the monuments of ancient Egypt the *pyramids* are the most famous. The accuracy of alignment, measurement and right angles of the pyramids is impressive by any standards (cf. Watson, 1987, p. 56). How did the Egyptian artisans of the Old and Middle Kingdoms (2686 - 1782 B.C.) construct right angles with such a high precision?

The nineteenth century historian Moritz Cantor conjectured that they might have known that a triangle whose sides measure 3, 4 and 5 units is right-angled, and that they might have used this knowledge for laying out right angles with the aid of a string with 3+4+5 = 12 knots (Cantor, 1880, Vol. 1, p. 105, 106). So far, however, Cantor's conjecture has not been confirmed at all. [1]

Although today we are not aware of any Egyptian document that attests that the 'Theorem of Pythagoras' was known actually at that time in Egypt, it seems probable that Pythagoras learnt it there during his 22-year stay (Diop, 1981, p. 436, 479).

We are sure, however, that the Ancient Egyptians had some knowledge of so-called ***Pythagorean triplets***, that is, numerical triplets (a, b, c) with the property $a^2 + b^2 = c^2$. In a papyrus that dates from the Middle Kingdom, the first problem reads:

> "...the area of a square of 100 is equal to that of two smaller squares. The side of one square is one-half and one-quarter of

---

* Slightly adapted translation of Chapter 4 in (Gerdes, 1989a).

1 In Section 3.9 of ***Awakening of Geometrical Thought in Early Culture*** (English version 2003; original version in German 1985) the author formulated alternative hypotheses.

the side of the other. Let me know the sides of the unknown squares" (Gillings, 1972, p. 161).

If the unknown sides are called *y* and *x*, the problem is equivalent to the following system of algebraic equations:

$$x = \frac{3}{4}y$$

$$x^2 + y^2 = 100.$$

The solution of this system is:

$$x = 6;\ y = 8\ \text{e}\ z = 10,$$

where *z* denotes the side of the given square. The Pythagorean triplet (6, 8, 10) is twice the famous triplet (3, 4, 5). As the papyrus does not mention a right-angled triangle with sides of 6, 8 and 10 units, the text does not imply knowledge of the Pythagorean proposition.

Is it possible that other sources of information, different from written texts, that might attest that the Pythagorean proposition was known in Ancient Egypt, or that might suggest in which context this theorem might have been discovered exist?

In this chapter, I will formulate a (new) hypothesis. [2]

Spiral patterns

The spiral is one of the most important elements of ancient Egyptian decoration. Since the V^{th} dynasty (2498 - 2345 B.C.) it appears as a detached ornament for small surfaces like scarabs (see Figure 1.1). Used as a continuous decoration on walls and furniture, it is still rare in the XII^{th} dynasty (1991 - 1782 B.C.), it becomes more frequent in the $XVIII^{th}$ (1551 - 1305 B.C.), flourishes in the XIX^{th} and $XXth^{th}$ (1293 -1070 B.C.), and is decadent in the $XXVI^{th}$ dynasty (664 - 525 B.C.) (Petrie, 1920).

2 See also Section 5.3 of *Awakening of Geometrical Thought in Early Culture*.

Scarabs
Figure 1.1

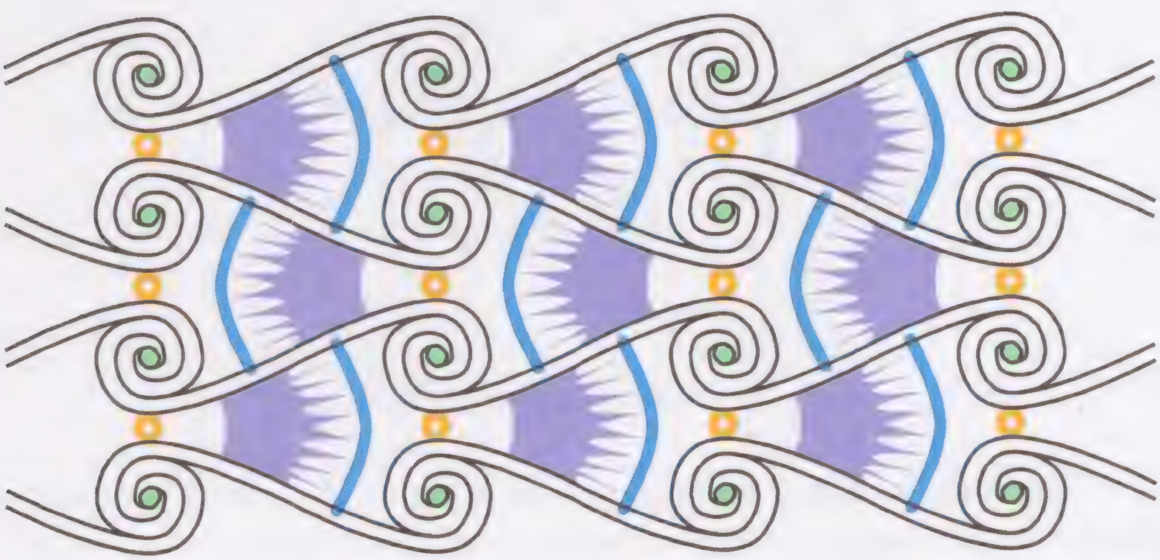


Figure 1.2

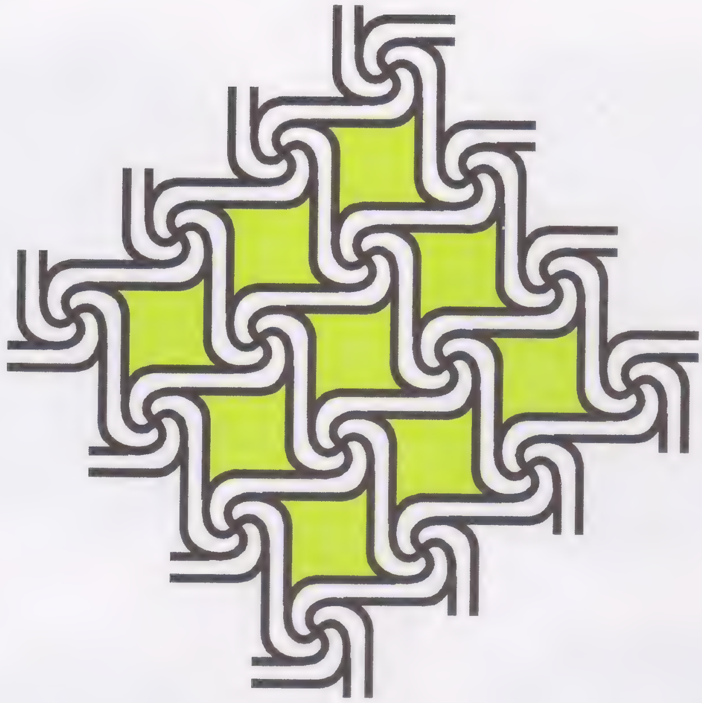


Figure 1.3

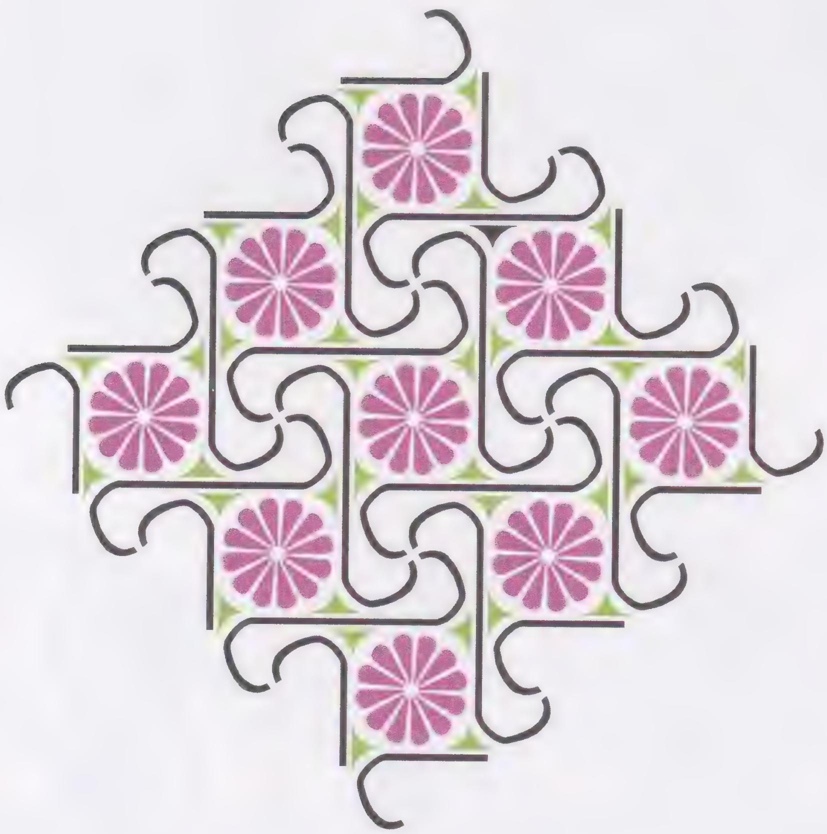


Figure 1.4

The simplest plane designs show a chequered pattern series of S-spirals or continuous lines of double spirals that were placed side by side (see Figure 1.2). Far more complex – "the glory of Egyptian line

decoration," as Petrie called it (1920, p.31) – is the quadruple spiral pattern, that constitutes a truly two-dimensional solution of the surface decoration problem (see Figure 1.3). Normally rosettes or lotus flowers were used to fill the hollow squares (see the example in Figure 1.4).

Construction of the quadruple spiral pattern

In order to draw the quadruple spiral pattern, the artisan had, probably, first of all, to lay out a grid, made out of **A'**- and **B'**-squares and to indicate the centres of the **A'**-squares (see Figure 1.5). Or, alternatively, the artisan had to mark a tiling with **A'**- and **B'**-squares on an already drawn square grid (see Figure 1.6). The last construction is easy when the lengths (a and b) of the **A'**- and **B'**-squares are integer multiples of the side of the unit square. The centres of the **A'**-squares, i.e. the centres of the spirals, generate an array of new squares **C'**, with side c (see Figure 1.7). Also, the centres of the **B'**-squares, i.e. generate a net of new squares, congruent to the **C'**-squares (see Figure 1.8). What relation will exist between the areas of the **A'**-, **B'**- and **C'**-squares?

Figure 1.5

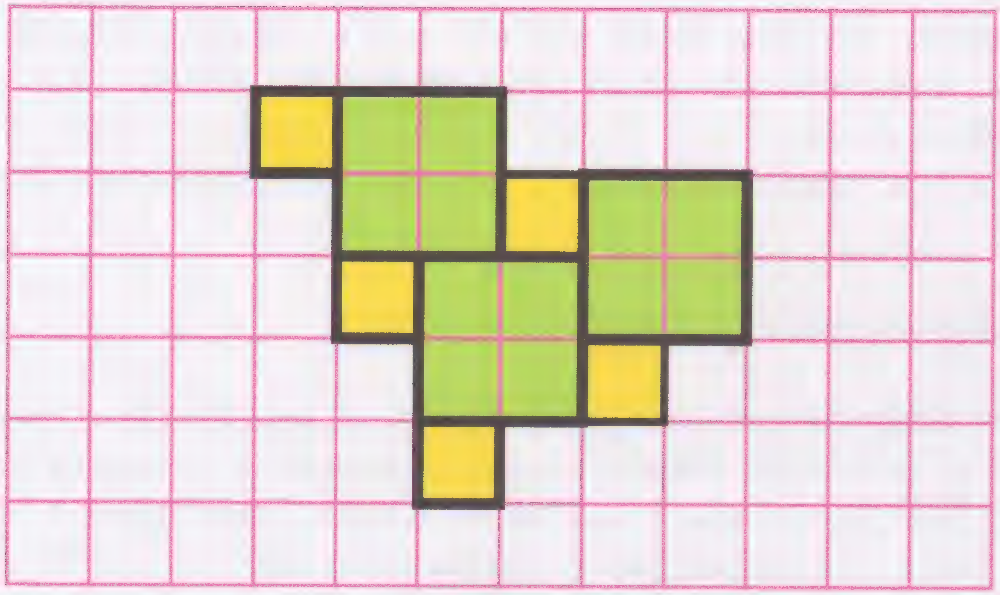


Figure 1.6

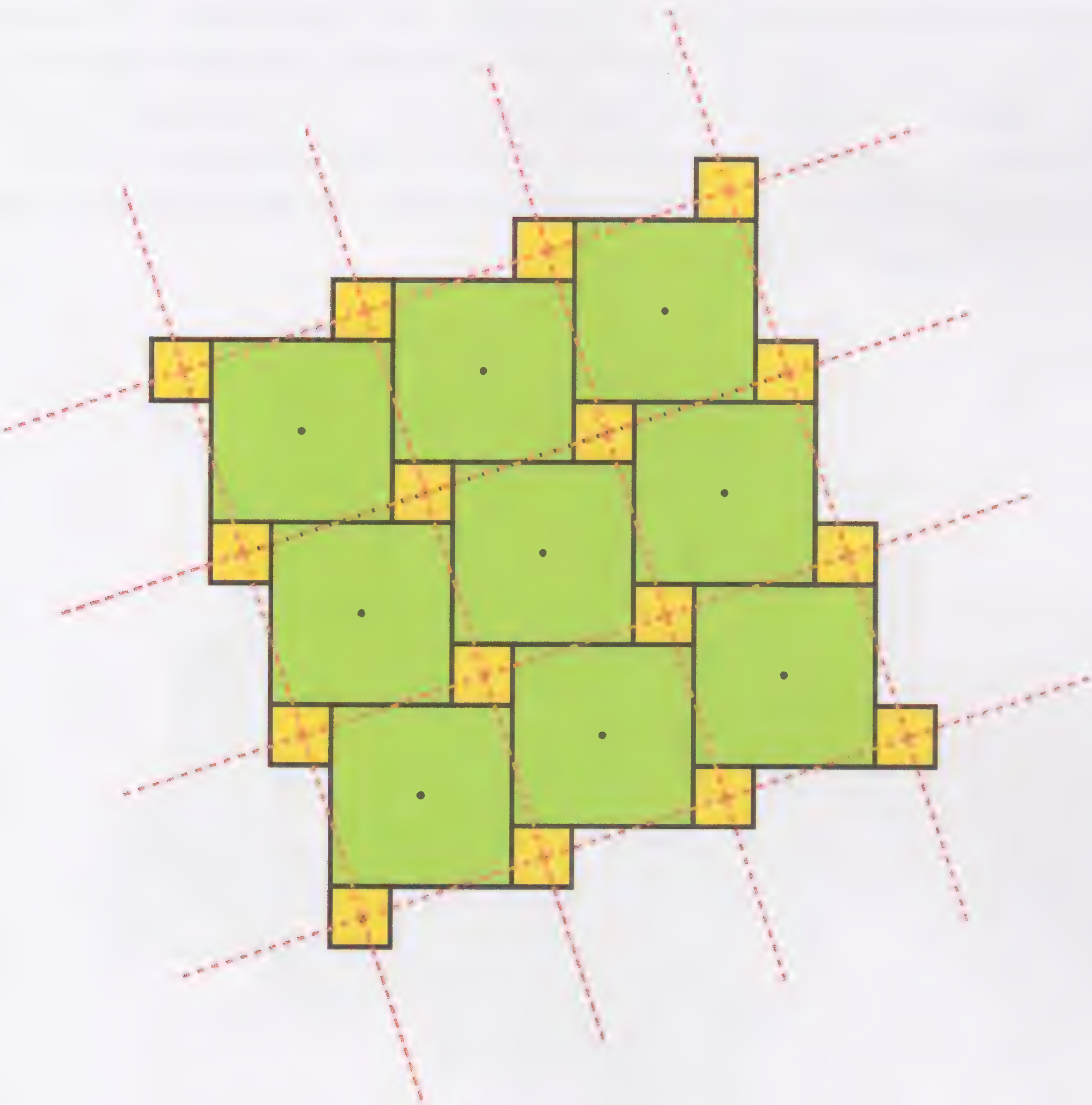


Figure 1.7

Figure 1.8

Equality of areas

As the **A'*B'**-grid may be considered as composed of joint **A'*B'-elements** of the type illustrated in Figure 1.9, there are as many **A'**- as **B'**-squares. As the centres of the **C'**- and **B'**-squares coincide, there are also as many **C'**- squares as there are **B'**-squares. We may be tempted to conclude therefore:

$$\mathbf{A + B = C},$$

where **A**, **B** and **C** denote the areas of the **A'**-, **B'**- and **C'**-squares.

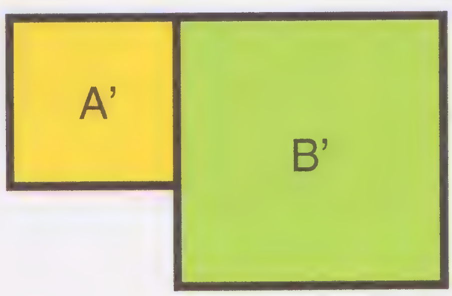


Figure 1.9

Theorem of Pythagoras? Theorem of Pappus?

Ancient Egyptian artisans constructed (the vertices of) squares that are equal in area to the sum of two given squares, during a period of more than a thousand years. It may be that during such a long period at least some artisans or observers of their work noticed that

$$\mathbf{A + B = C}.$$

From this observation, it is only a short step to the Pythagorean Theorem, as Figure 1.10 suggests: From one vertex of a **C'**-square to the next, i.e. from one centre of a spiral to a neighbour centre, we may move first a distance *b* horizontally to the right and then a distance *a* vertically upwards. In other words, a triangle with sides *a*, *b* and *c* is right-angled, and

$$\mathbf{A + B = C}$$

implies

$$\mathbf{a^2 + b^2 = c^2}.$$

Therefore, although as yet no written sources have been found that show so, it is still more likely that the Pythagorean Theorem was known in Ancient Egypt.

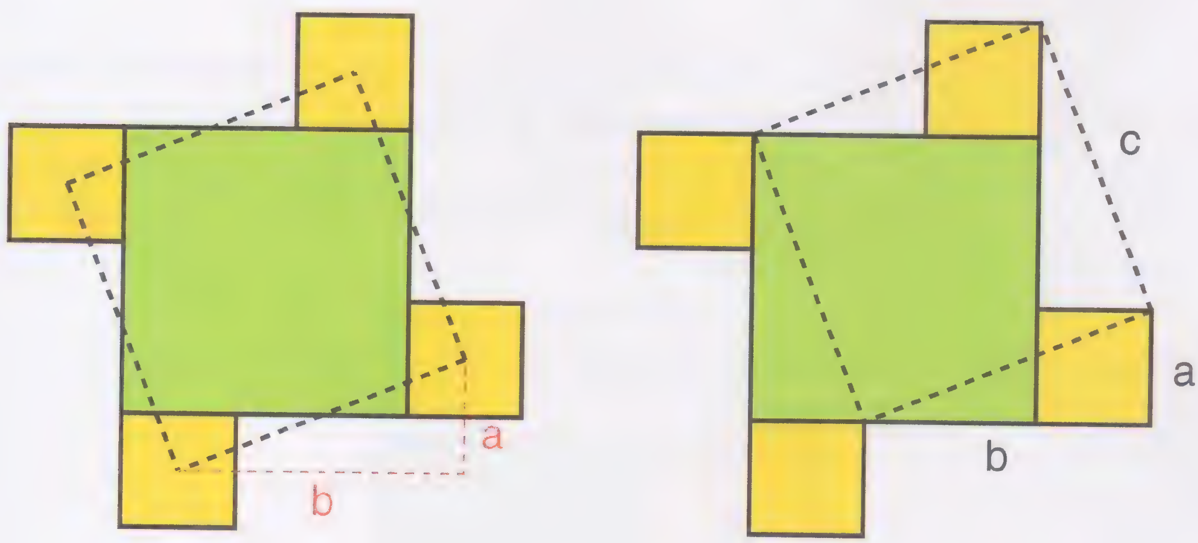


Figure 1.10

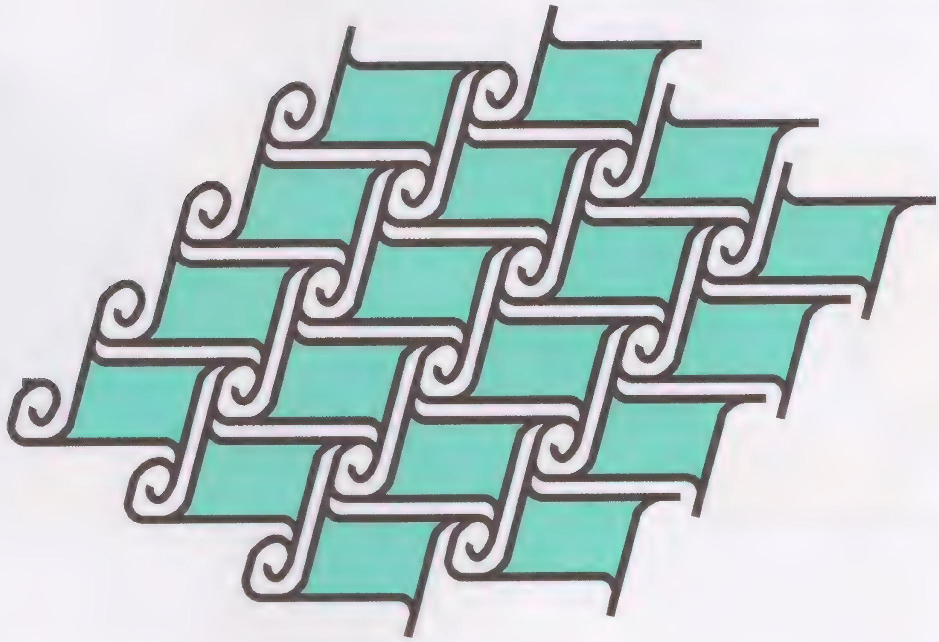


Figure 1.11

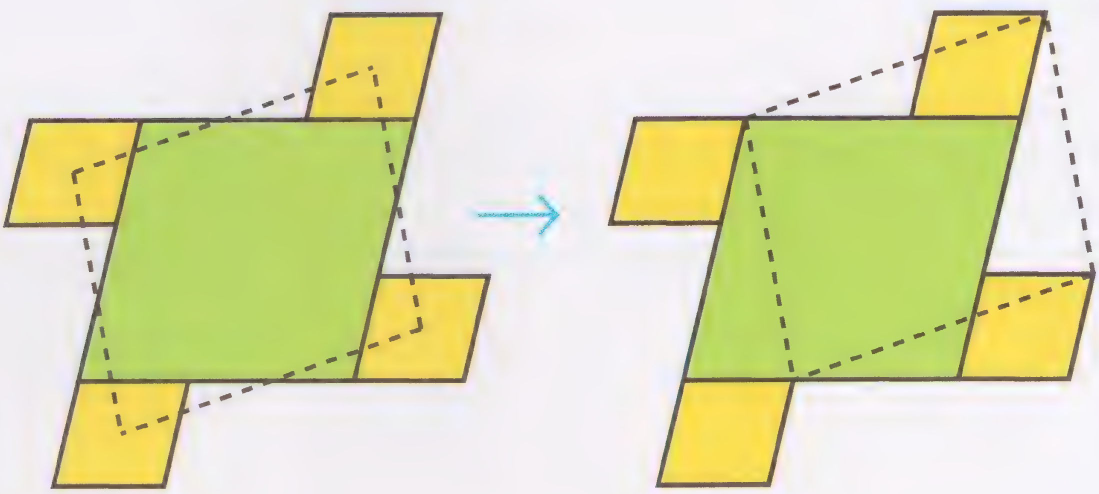
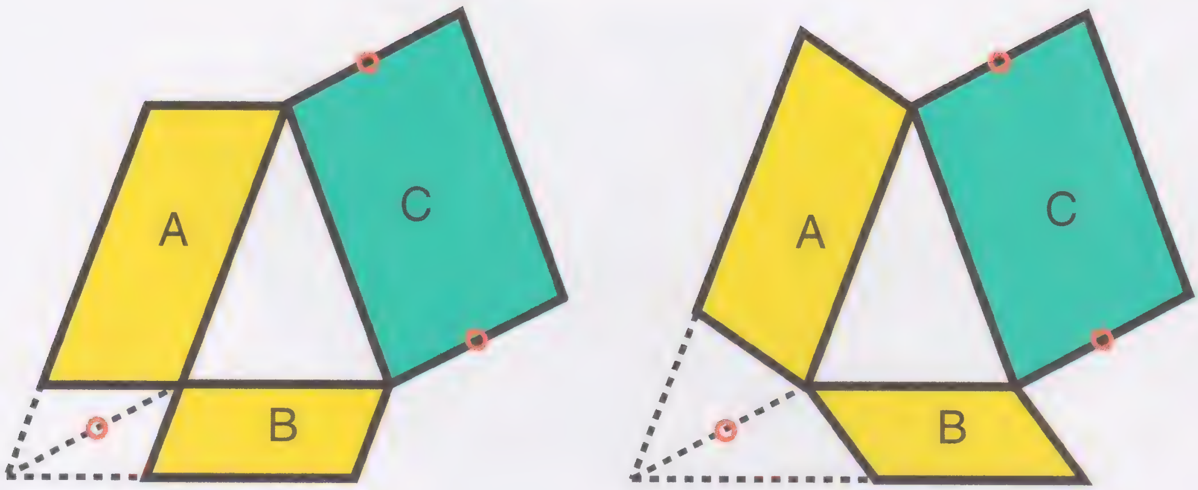


Figure 1.12

As the quadruple spiral pattern is "sometimes rhombic in plan" (Petrie, 1920, p. 32), Egyptian artisans knew (implicitly at least) how to construct a parallelogram equal in area to the sum of two given similar parallelograms (see Figure 1.11) and an *extension* of the Pythagorean proposition for parallelograms (see Figure 1.12) might have been conjectured. From this extension it is only a short step to the generalization given by Pappus of Alexandria (Egypt, ca. 300 A.D.) in Book IV of his *Mathematical Collection* (see Figure 1.13). This raises a new question: where had Pappus got his inspiration or knowledge from? Perhaps from the work of artisans in Ancient Egypt?



A + B = C

A + B = C

Figure 1.13

Education

In the teaching of geometry, the grid that underlies the Egyptian quadruple spiral pattern and is composed of **A'**- and **B'**-squares (see figure 1.5) may serve as a starting point to arrive at the discovery and at a proof, both of the Pythagoras Proposition and of the Theorem of Pappus.

Chapter 2

FROM WOVEN BUTTONS TO THE THEOREM OF PYTHAGORAS *

By pulling a little lasso around a square-woven button (see Figure 2.1), it is possible to fasten the top of a basket, as is commonly done in southern parts of Mozambique. The square button is woven out of two strips. Figure 2.2 shows how the artisan starts to weave the button.

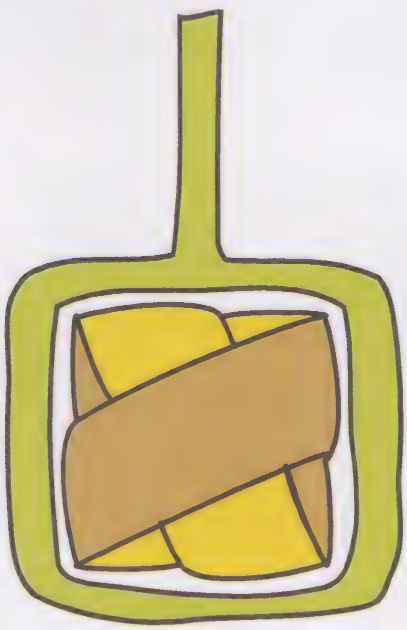


Figure 2.1

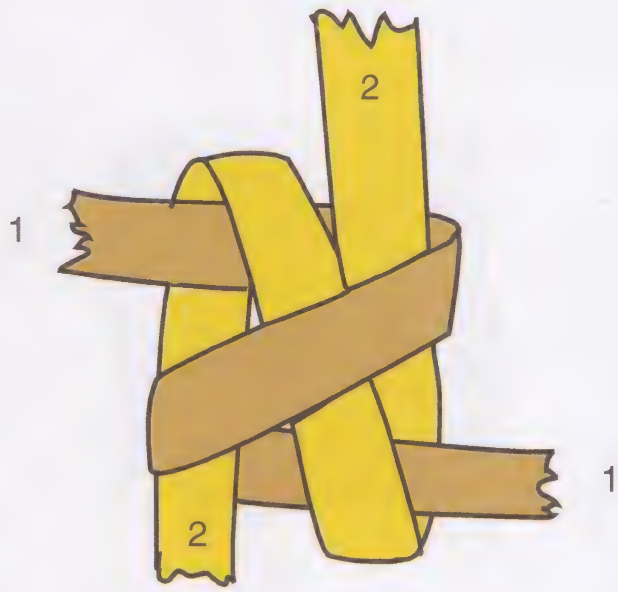


Figure 2.2

Figure 2.3 presents the front- and backside of the initial knot (Figure 2.2).

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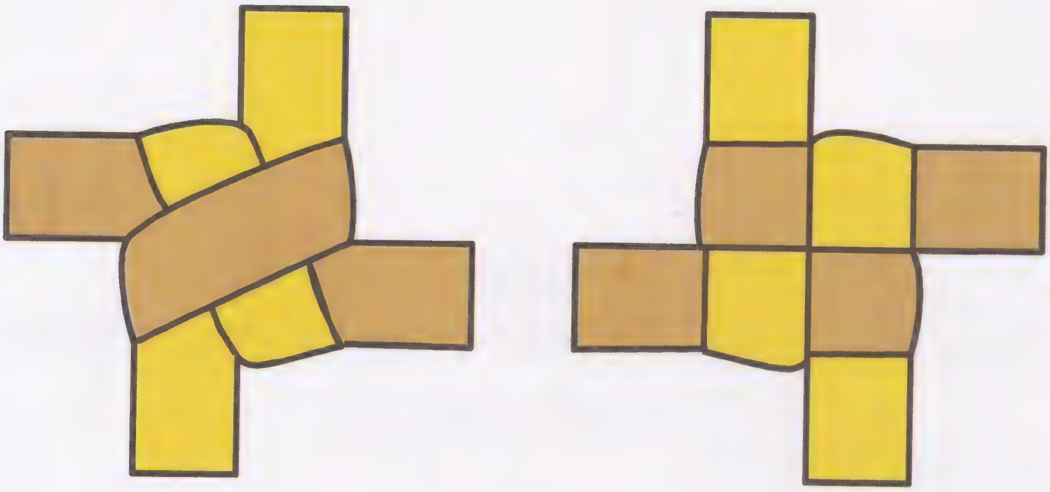
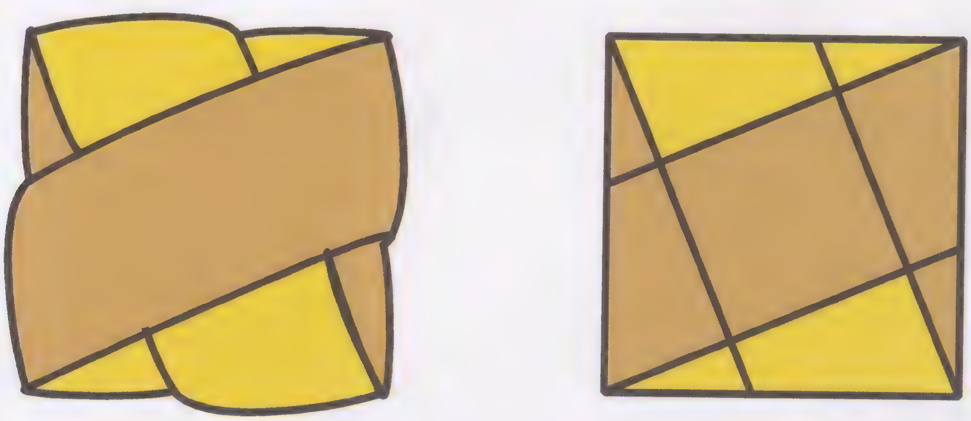
* Slightly adapted version of an example presented in (Gerdes, 1988d, 1990b), referring to a dialogue between the lecturer and students in the context of mathematics teacher education in Mozambique.



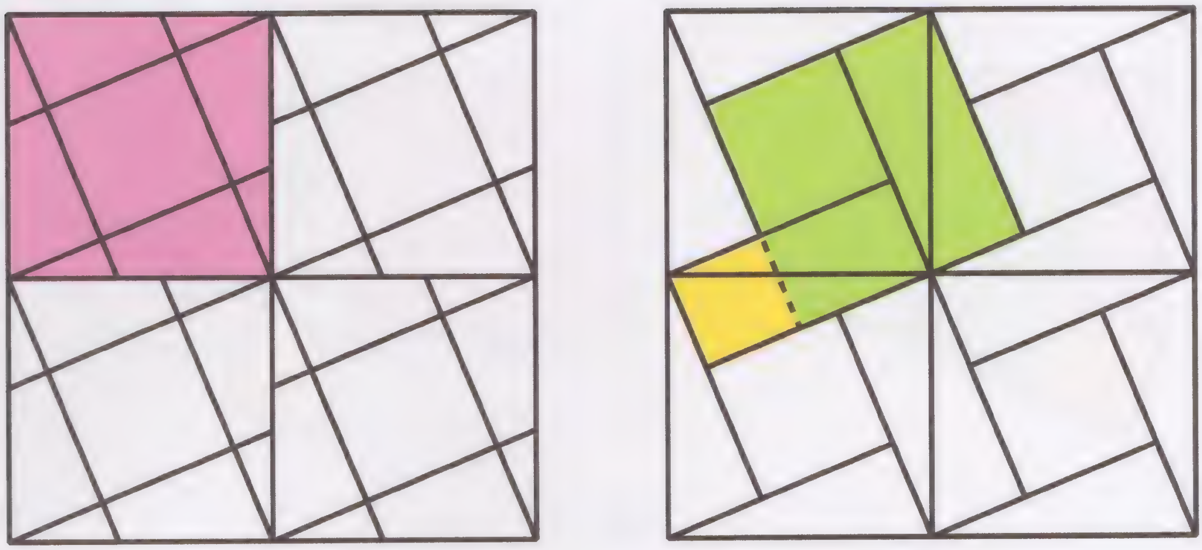
Front- and backside
Figure 2.3

When we consider the square-woven button from above, we observe the design displayed in Figure 2.4a. After rectifying the slightly curved lines and by making the hidden lines visible, we obtain the design illustrated in Figure 2.4b. In its middle a second square appears.



a b
Figure 2.4

What other squares may be observed when we join some of these square-woven buttons together? Do other figures with the same area as (the top of) a square-woven button (see Figure 2.5a) appear? Yes, if you like, you may extend some of the line segments or rub out some others (Figure 2.5b).



a b

Figure 2.5

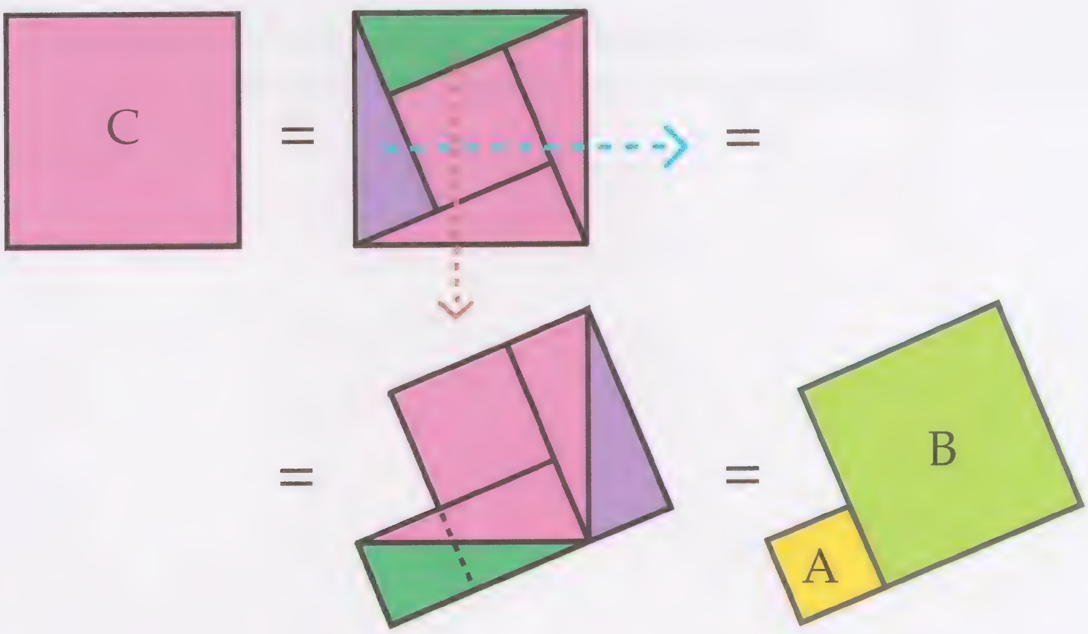


Figure 2.6

What do you observe? Equality in areas? (cf. Figure 2.6)

We may arrive at the conclusion that

$$\mathbf{C = A + B}$$

(see Figure 2.7), that is, we arrive at the so-called Theorem of Pythagoras.

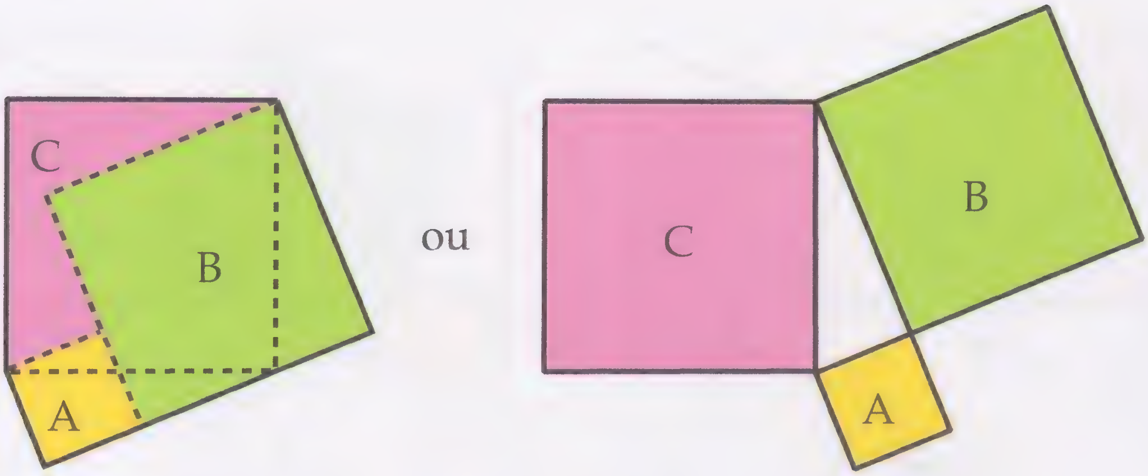


Figure 2.7

My teacher-students themselves rediscovered this important theorem and succeeded in proving it. One of them remarked enthusiastically: "Had Pythagoras - or somebody else before him - not discovered this theorem, *we* would have discovered it"...

Chapter 3

FROM FOURFOLD SYMMETRY TO 'PYTHAGORAS'

Designs or pattern details that display a fourfold symmetry, that is a rotational symmetry of 90^{o}, occur frequently in African decoration. Figure 3.1 displays some examples. In this chapter we will explore the fact that points, which correspond to each other under fourfold symmetry always constitute the vertices of a square.

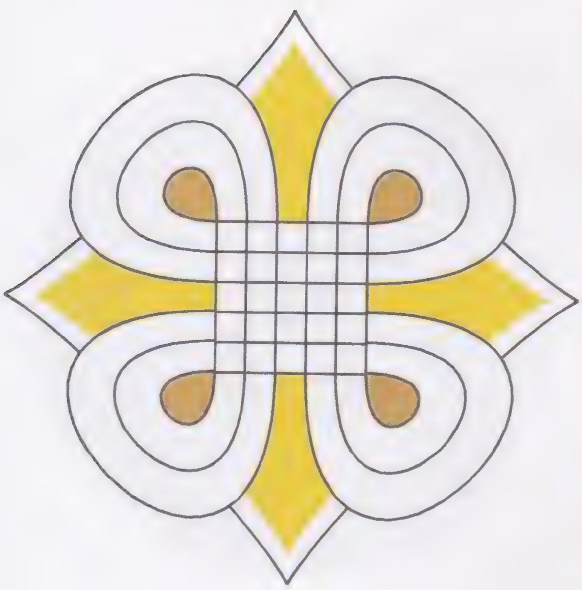
We will take an object from Mozambique as starting point. Any one of the designs presented in Figure 3.1 may serve as an alternative starting point.

a

b

Designs painted on pre-dynastic vases
(Middle 4th millennium B.C., Ancient Egypt)
Figure 3.1

Design painted on a vase; centre with fourfold symmetry
(Thebes, Ancient Egypt, c.1350 B.C.)
c



Design painted on textile
(Ghana)
d

Decorative design on a mat from
the Lower-Congo area
e

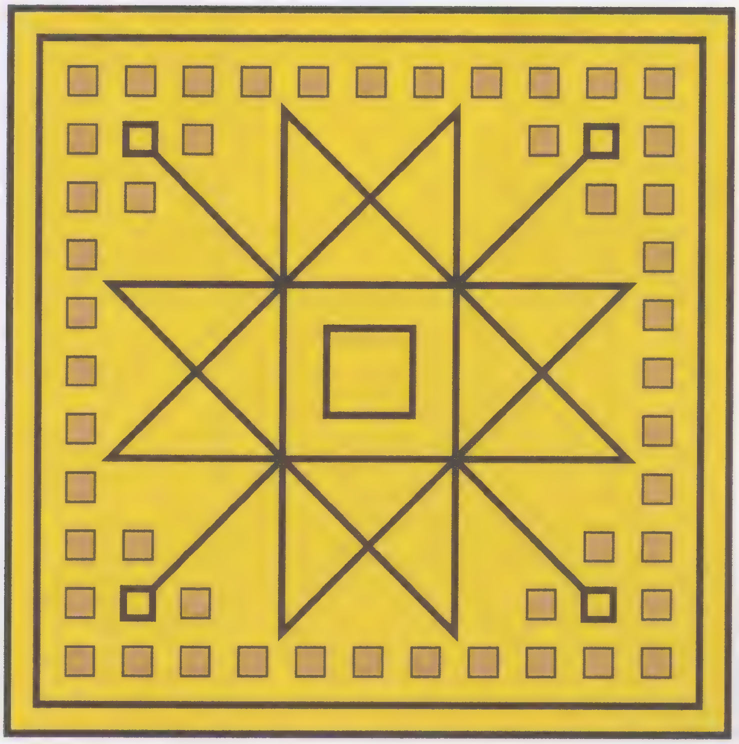
Figure 3.1 (continued)

Weaving pattern (Cokwe, Angola)
f

Bronze weight (Ashanti, Ghana)
g

Design on a mat
(Zanzibar, Tanzania)
h

Figure 3.1 (continued)



Textile design from the Upper-Senegal area

i

Examples of body stamps from the Igbo (Nigeria)

j

Mosaic design (Madagascar; Lesotho)

k

Figure 3.1 (continued)

Adinkra stamps of the Ashanti (Ghana)

l

Motif engraved on wooden doors (Yoruba, Nigeria)

m

Detail of a design on plaited raffia clothes (Kuba, Congo)

n

Decorative motif from Angola

o

Letter of the Njoya alphabet (Cameroon)

p

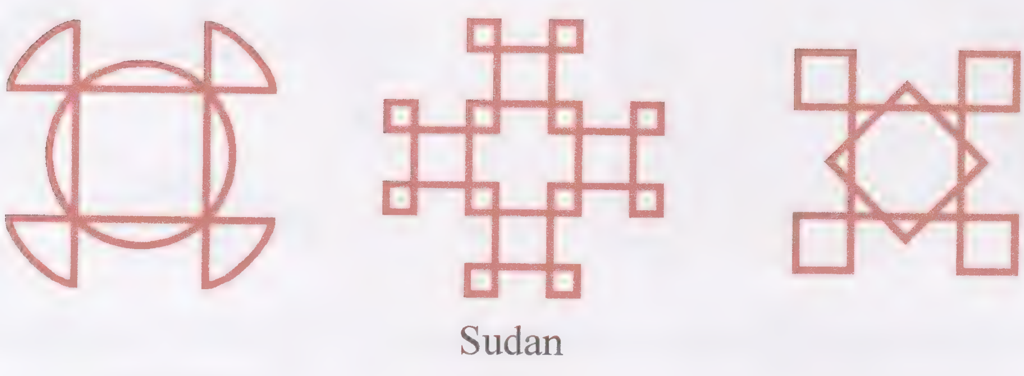
Figure 3.1 (continued)

Benin
q

Madagascar
r

Congo
s

Angola
t



Sudan
u

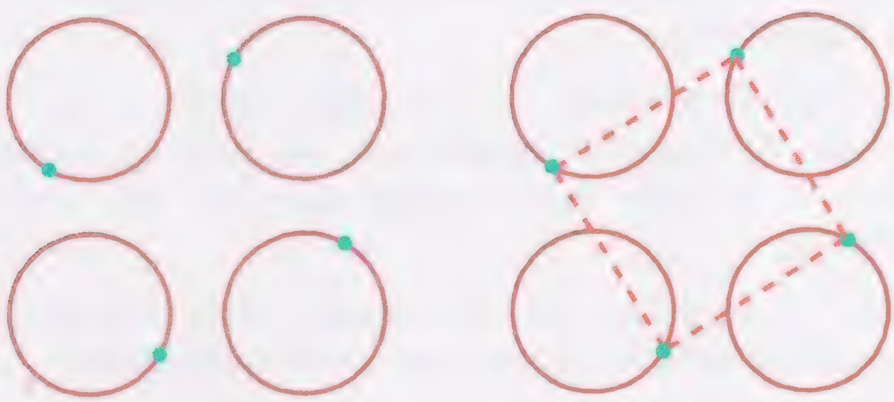
Decorative motifs with fourfold symmetry
Figure 3.1 (conclusion)

3.1 *Decoration of a headrest*

Figure 3.2 shows an old headrest from the Maputo region in the south of Mozambique. It displays a decoration with fourfold symmetry.

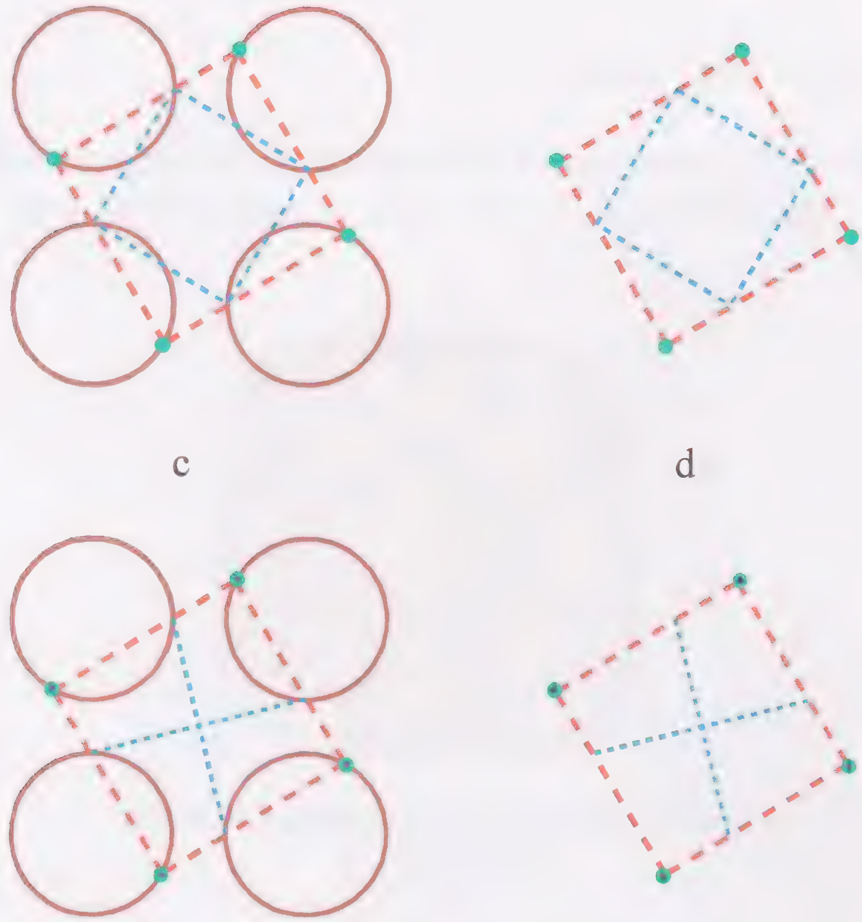
Decoration of a Mozambican headrest
Figure 3.2

When we link four corresponding points on the circles (see the example in Figure 3.3a) by straight-line segments, we obtain a square (Figure 3.3b). The corresponding points of intersection of these segments with the circles are the vertices of a square too (Figure 3.3c). Such a square is inscribed in the first square (Figure 3.3d).



a b

Figure 3.3



c d

e f

Figure 3.3

If, instead of linking the four neighbouring points of intersection as in Figure 3.3c, we link the opposed intersection points as in Figure 3.3e, we obtain a cross that divides the first square into four congruent parts (see Figure 3.3f).

Both designs produced in this way (Figures 3.3d, 3.3f), lead easily to the Pythagorean proposition, as will be shown in the following. It is with this meaning that we call both designs ***pythagorasable***.

Figure 3.4 presents a second example. This time the centres of the four circles have been selected as corresponding points.

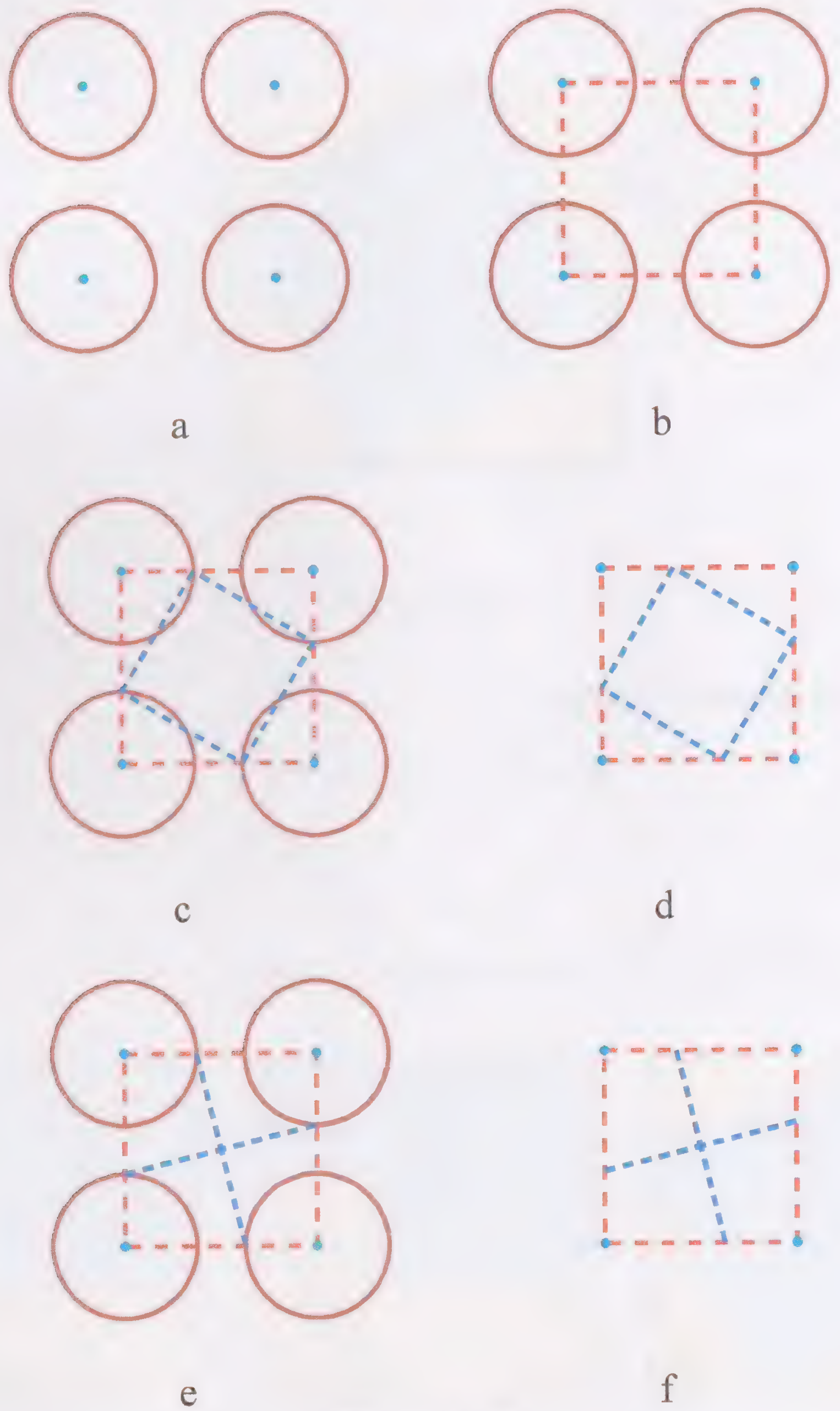


Figure 3.4

Let *a*, *b* and *c* denote the sides of the congruent right-angled triangles in Figure 3.5 (Cf. Figures 3.3d and 3.4d). As the side of the large square measures a+b, its area is $\mathbf{(a+b)^2}$. The area of this square is equal to the sum of the areas of the four right-angled triangles (four times half **ab**) plus the area of the inscribed square ($\mathbf{c^2}$). Therefore

$$(a+b)^2 = 2ab + c^2.$$

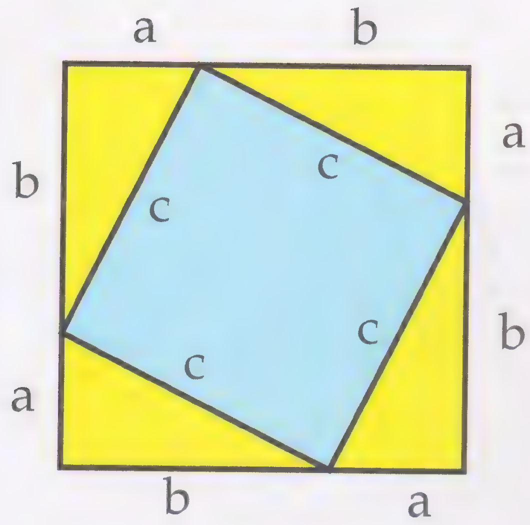


Figure 3.5

Taking into account the equality

$$(a+b)^2 = 2ab + a^2 + b^2,$$

we find

$$a^2 + b^2 = c^2,$$

that is, we arrive at the Pythagorean proposition.

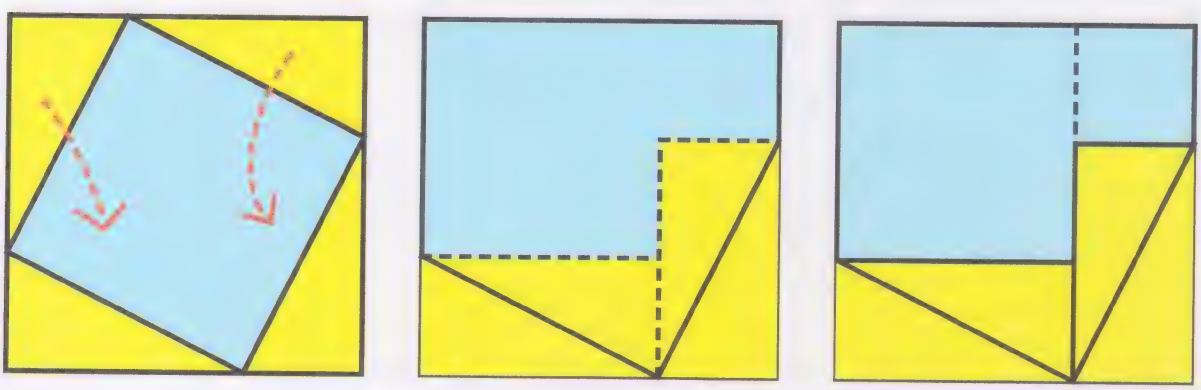


Figure 3.6

Geometrically we may arrive at this result in other ways too. When we move the upper right-angled triangle downwards as in Figure 3.6, we observe that the large square may be considered as composed of one square with side *a*, one square with side *b* and four right-angled triangles with sides *a*, *b* and *c*. Initially the same large square was composed of a square of side *c* circumscribed by four right-angled triangles with sides *a*, *b* and *c*. Therefore, the area of the

square with side ***c*** is equal to the sum of the areas of the squares with sides ***a*** and ***b***, or

$$c^2 = a^2 + b^2.$$

Figure 3.7 shows an alternative and well-known decomposition of the big square with side a+b, that, according to some authors (Van der Waerden, 1983; Mainzer, 1980), might have been used by Pythagoras himself to prove 'his' proposition (cf. Loomis, 1972, p. 49, 50, 154, 223, 229).

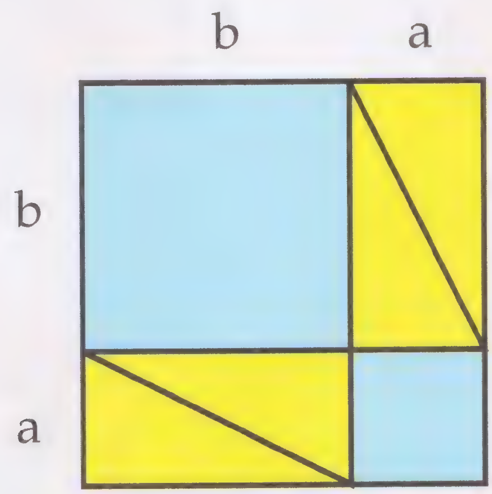


Figure 3.7

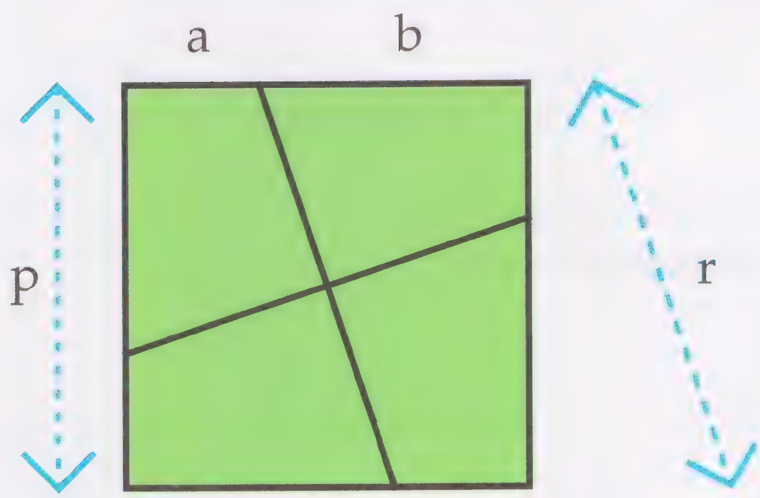


Figure 3.8

Let us now return to Figure 3.3f and 3.4f. Let ***p*** be the side of the square and ***r*** the length of an arm of the cross (see Figure 3.8). We may join the four congruent quadrilaterals in such a way that a new

and larger square with side *r* appears (see Figure 3.9). At its centre a square hole appears. Let *q* be its side. From the construction of the new square follows immediately:

$$r^2 = p^2 + q^2.$$

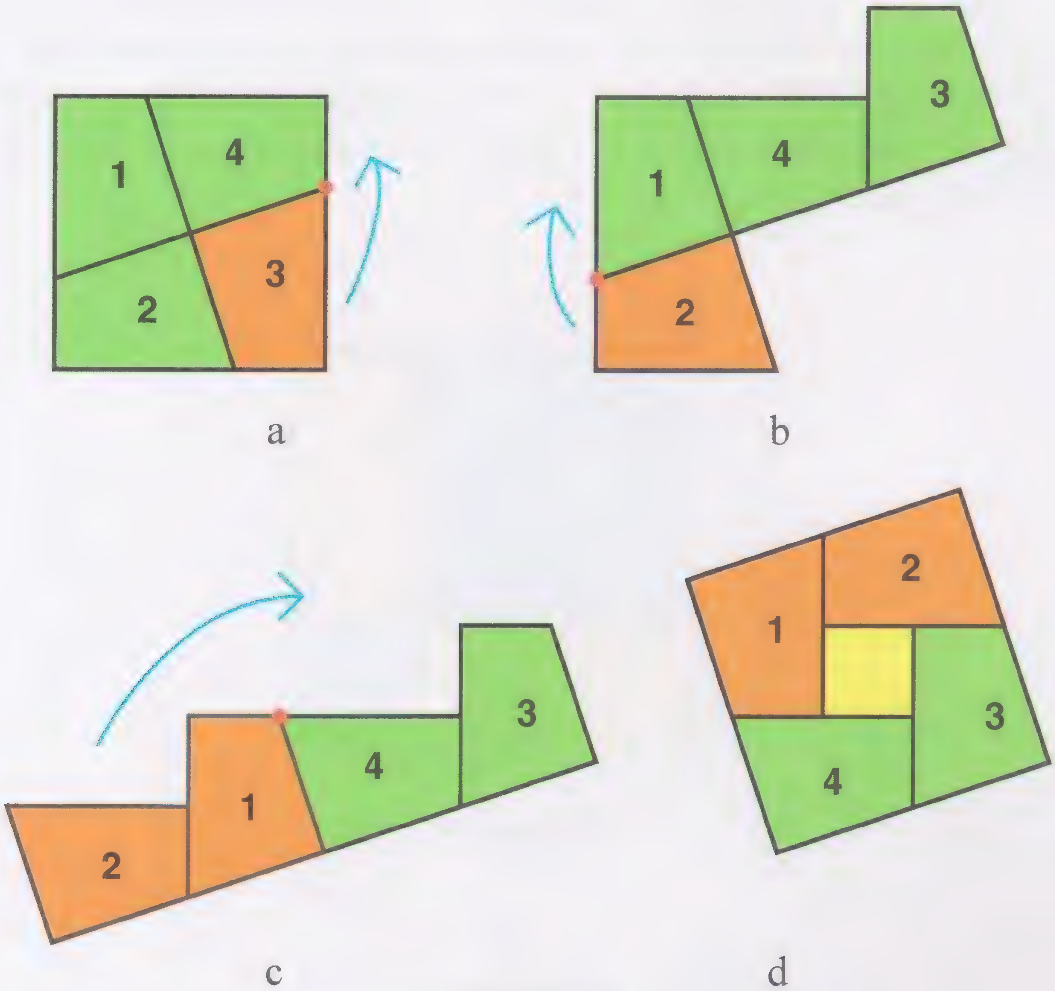


Figure 3.9

As q = b-a, we see (see Figure 3.10) that *p*, *q* and *r* constitute the sides of a right-angled triangle.

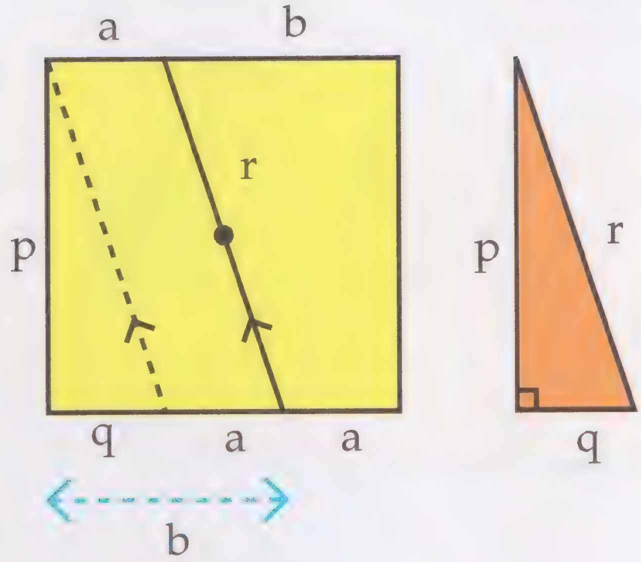


Figure 3.10

This reasoning may be used to arrive at a so-called 'dissection' proof for the Pythagorean proposition.

Consider an arbitrary right-angled triangle with sides *p*, *q* and *r*. Let *p* be larger than *q*. Draw squares on the legs. Through the centre of the square of side *p*, construct a cross with one of its arms parallel to the hypotenuse of the right-angled triangle (see Figure 3.11).

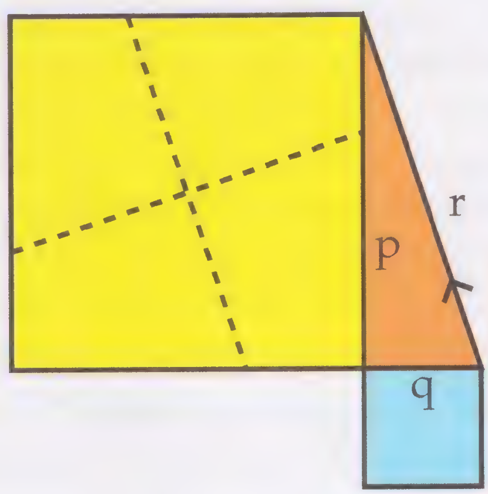


Figure 3.11

As seen before, the four pieces into which the *p*-square has been dissected may be joined together with the *q*-square to obtain the *r*-square (see Figure 3.12). Therefore:

$$r^2 = p^2 + q^2.$$

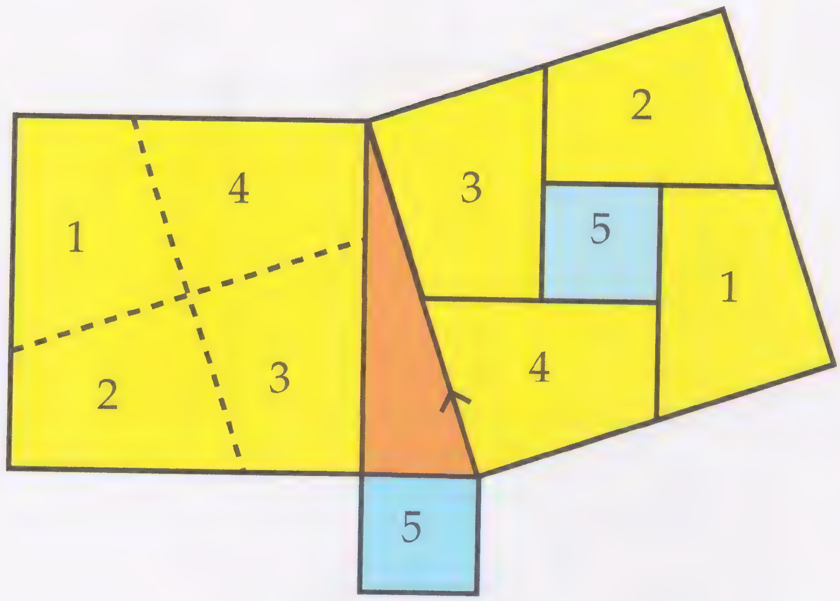


Figure 3.12

This proof was found – for the first time (?!) – by Henri Perigal as late as 1873 (Cf. Loomis, 1972, p. 104).

3.2 Cokwe sand drawings

When the Cokwe (Angola) met at their central village places or at their hunting camps, sitting around a fire or in the shadow of leafy trees, they used to spend their time in conversations, that were illustrated by standardised drawings on the ground. Their ***akwa kuta sona*** – drawing experts – invented an interesting mnemonic device to memorise the drawings. After cleaning and smoothing the ground, they first set out with their fingertips an orthogonal net of equidistant points. Many times they added a second net in such a way that the points of the second one are the centres of the unit squares of the first. Around the dots of the drawers trace their figures. For instance, in order to draw a tortoise, they superimposed a 2^2-point square and a 3^2-point-square (see Figure 3.13). To draw an ox-stall, we have to join a 3^2-point-square with a 4^2-point-square, as Figure 3.14 illustrates.

A Pythagorean triplet [1]

The Cokwe used a 5 point reference frame to draw their characteristic ***cingelyengenlye*** motif (see Figure 3.15a and b), a very old design that has already appeared on rock paintings in the Upper

1 Example given in (Gerdes, 1988c).

Zambeze region (Redinha, 1948). A 5-point grid was also used to draw some other patterns (see Figure 3.15c, d and e).

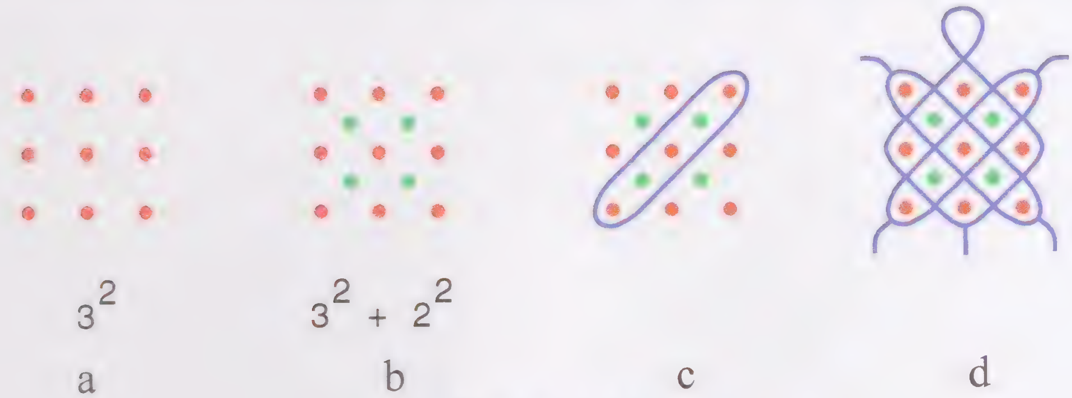


Figure 3.13

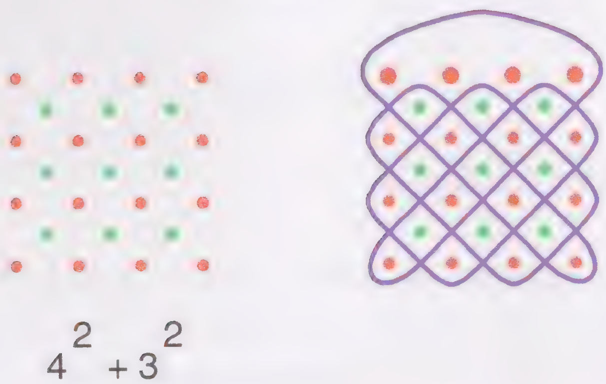


Figure 3.14

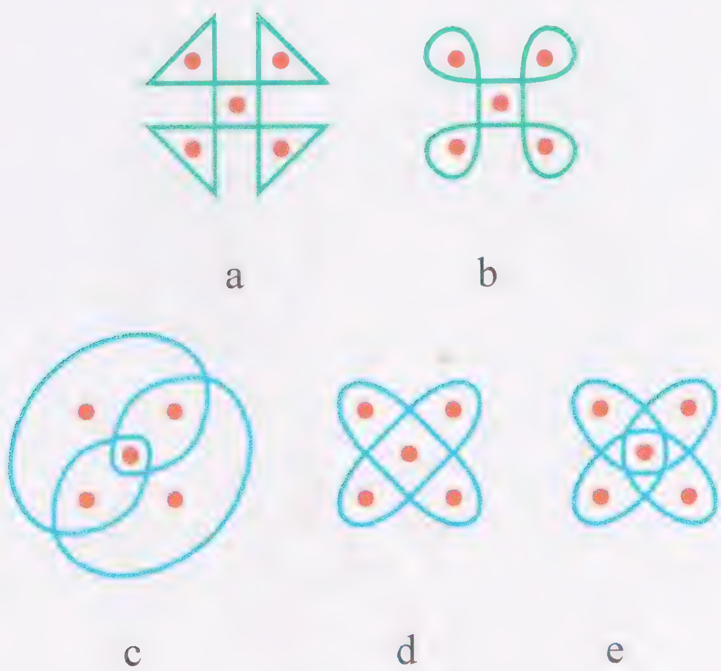


Figure 3.15

By trying to cover the reference frames necessary for the tortoise, ox-stall and other designs with these five point patterns, pupils may discover (see Figure 3.16) the Pythagorean triplet (3, 4, 5):

$$3^2 + 4^2 = 5^2.$$

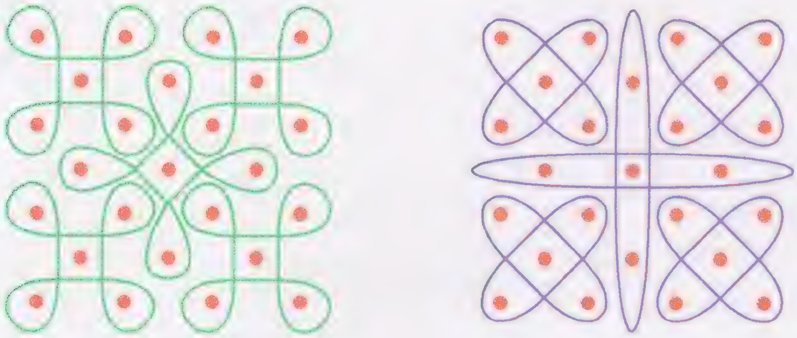


Figure 3.16

Easy ways towards Pythagoras

Figure 3.17 shows Cokwe sand drawings with fourfold symmetry. In all cases, it is easy to move from them to one of the aforementioned ***Pythagorasable*** designs in Figure 3.18.

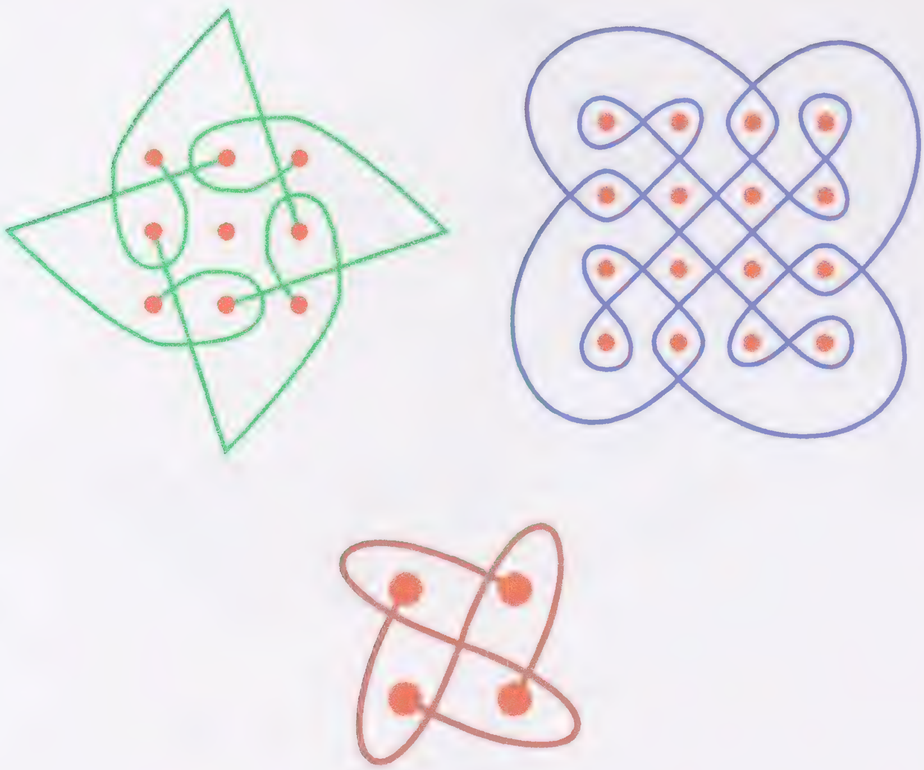


Figure 3.17

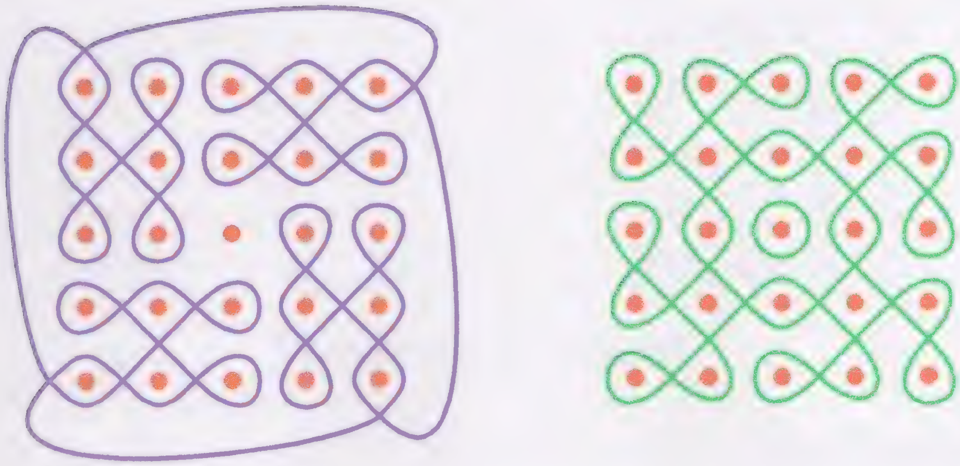


Figure 3.17 (conclusion)

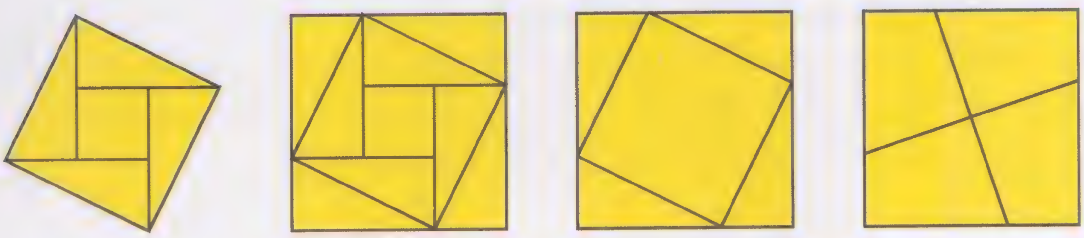


Figure 3.18

To achieve this, it is sufficient to consider corresponding points of the nets of dots. Their correspondence becomes visible when we draw the designs. Figure 3.19 resumes the idea.

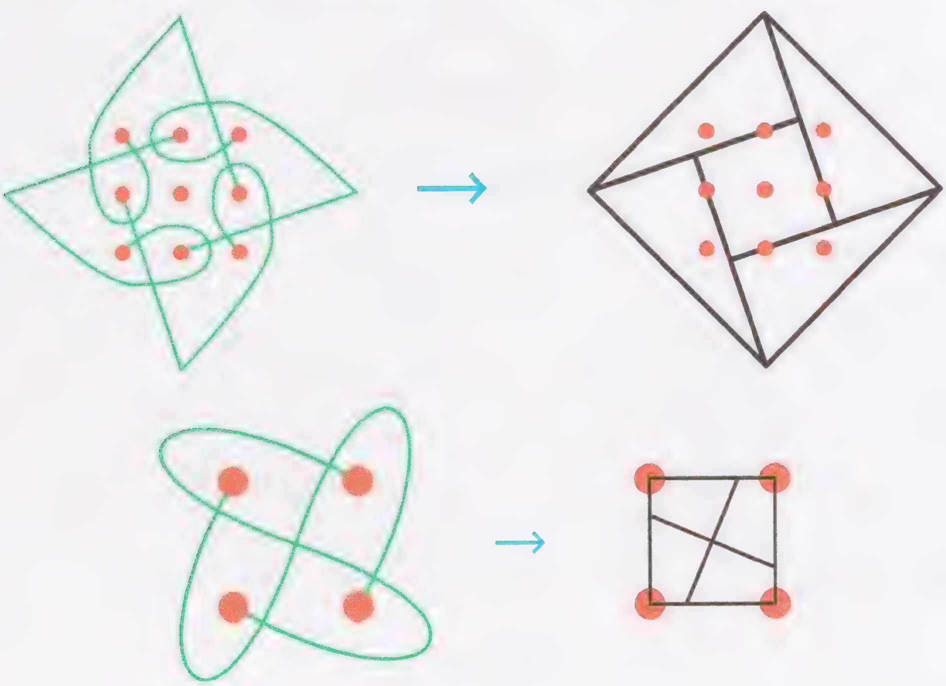


Figure 3.19

Figure 3.19 (conclusion)

A widespread cross

Figure 3.20 displays a cross that may be encountered as a weaving design, as a wall decoration and as a tattoo in Cameroon, Ivory Coast, Congo and Angola among the Cokwe (cf. Mveng, 1980, p.120). When we link four corresponding vertices, we easily obtain a ***Pythagorasable*** design, as Figure 3.21 illustrates.

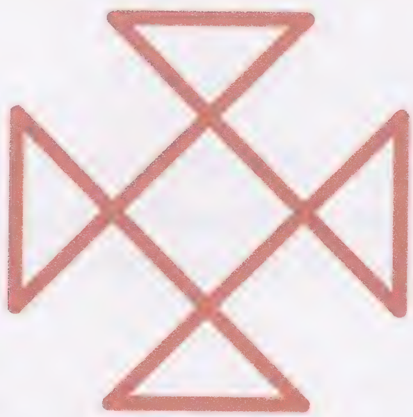


Figure 3.20

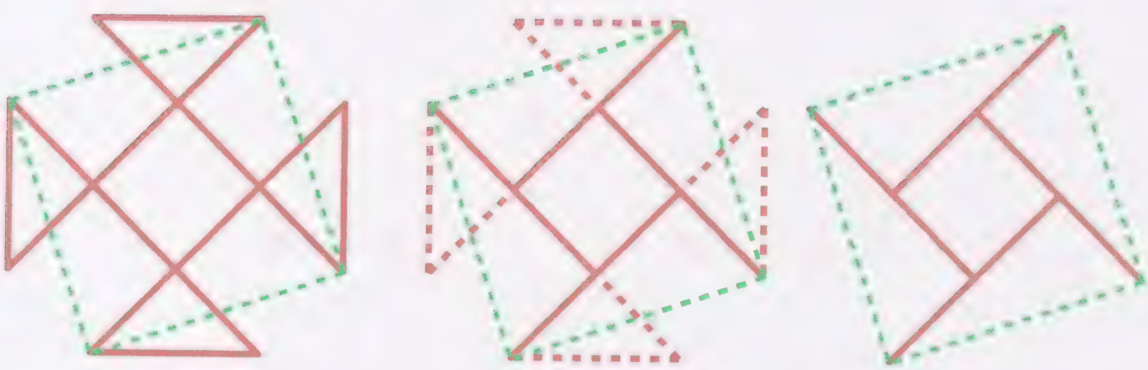


Figure 3.21

3.3 *A beautiful Kuba tiling*

Figure 3.22 illustrates an interesting tessellation of the plane composed of 'hooks' and squares. This pattern appears traditionally on woven mats and on embroidered raffia textiles of the (Bu)Shongo or (Ba)Kuba (Congo), and is called *Mikope Ngoma*, that is, the drums of king Mikope (Meurant, p. 204; Torday, p. 203).

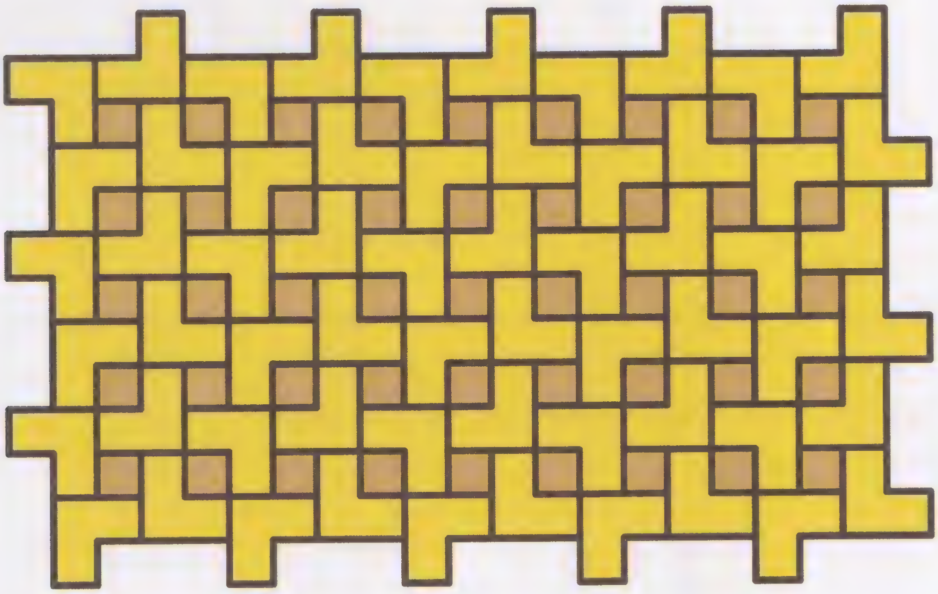
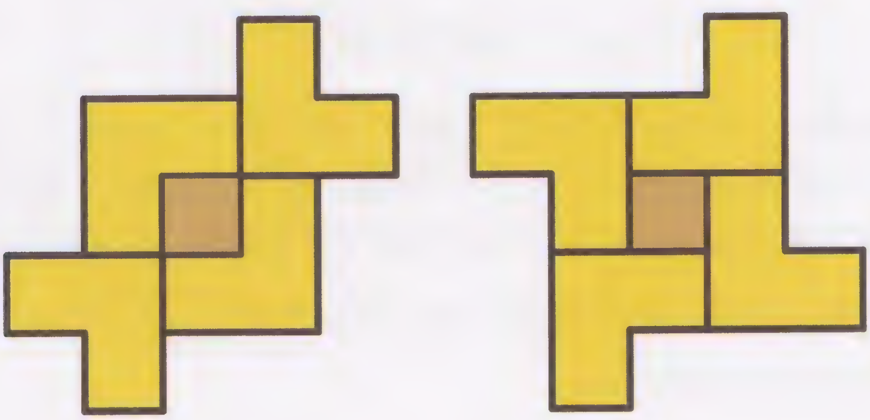


Figure 3.22

Each square is embraced by four hooks, as shown in Figures 3.23a and b. The second design (Figure 3.23b) displays a fourfold symmetry. When we link the corresponding vertices as indicated in Figure 3.24, we obtain a square that has the same area as the initial design.



a b

Figure 3.23

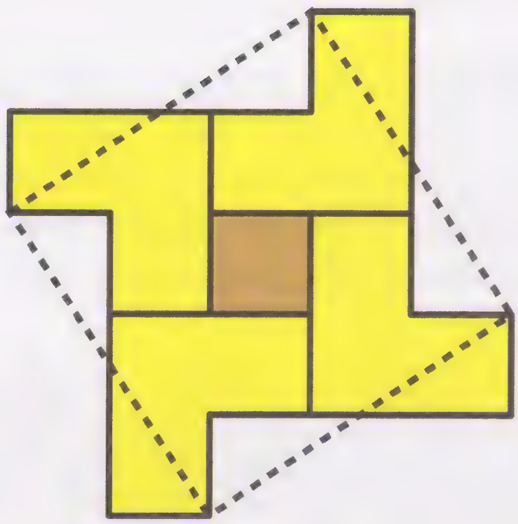


Figure 3.24

We may generalise this design in such a way (see Figure 3.25a) that the new square still has the same area as the design (see Figure 3.25b) and that the corresponding right-angled triangle is arbitrary with sides ***a***, ***b*** and ***c***.

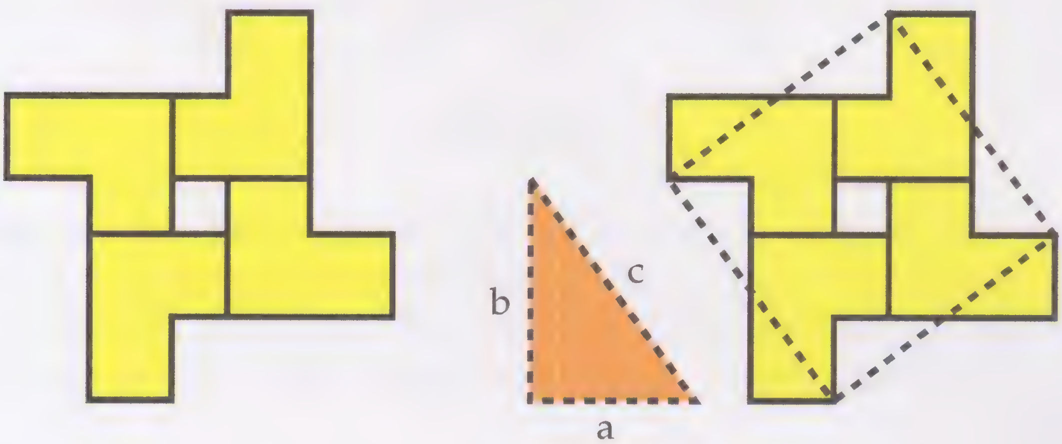


Figure 3.25

Therefore, we find (see Figure 3.26) that the area of the large square (c^2) is equal to the area of the small square plus two times the area of the rectangle formed by joining the hooks. As the small square has side **b-a**, and the rectangles have sides ***a*** and $\frac{a}{2} + [(b-a) + \frac{a}{2}]$, that is ***b***, we may conclude that

$$c^2 = (b-a)^2 + 2ab = b^2 + a^2 .$$

In other words, we prove the Theorem of Pythagoras.

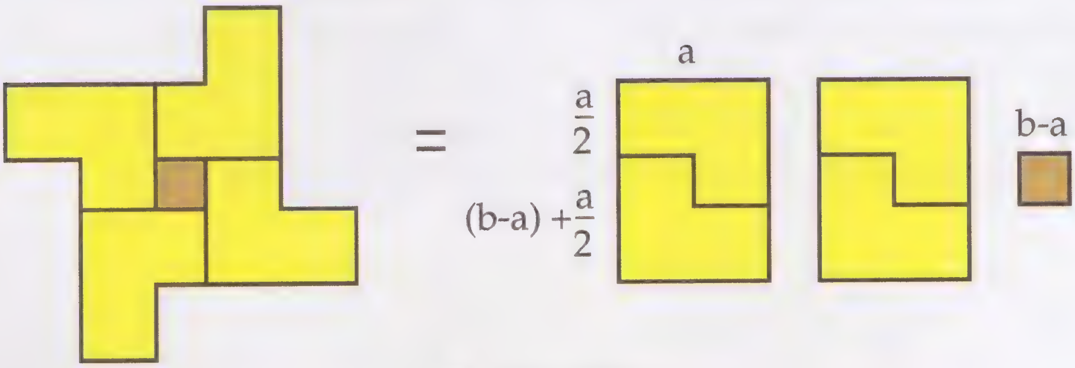


Figure 3.26

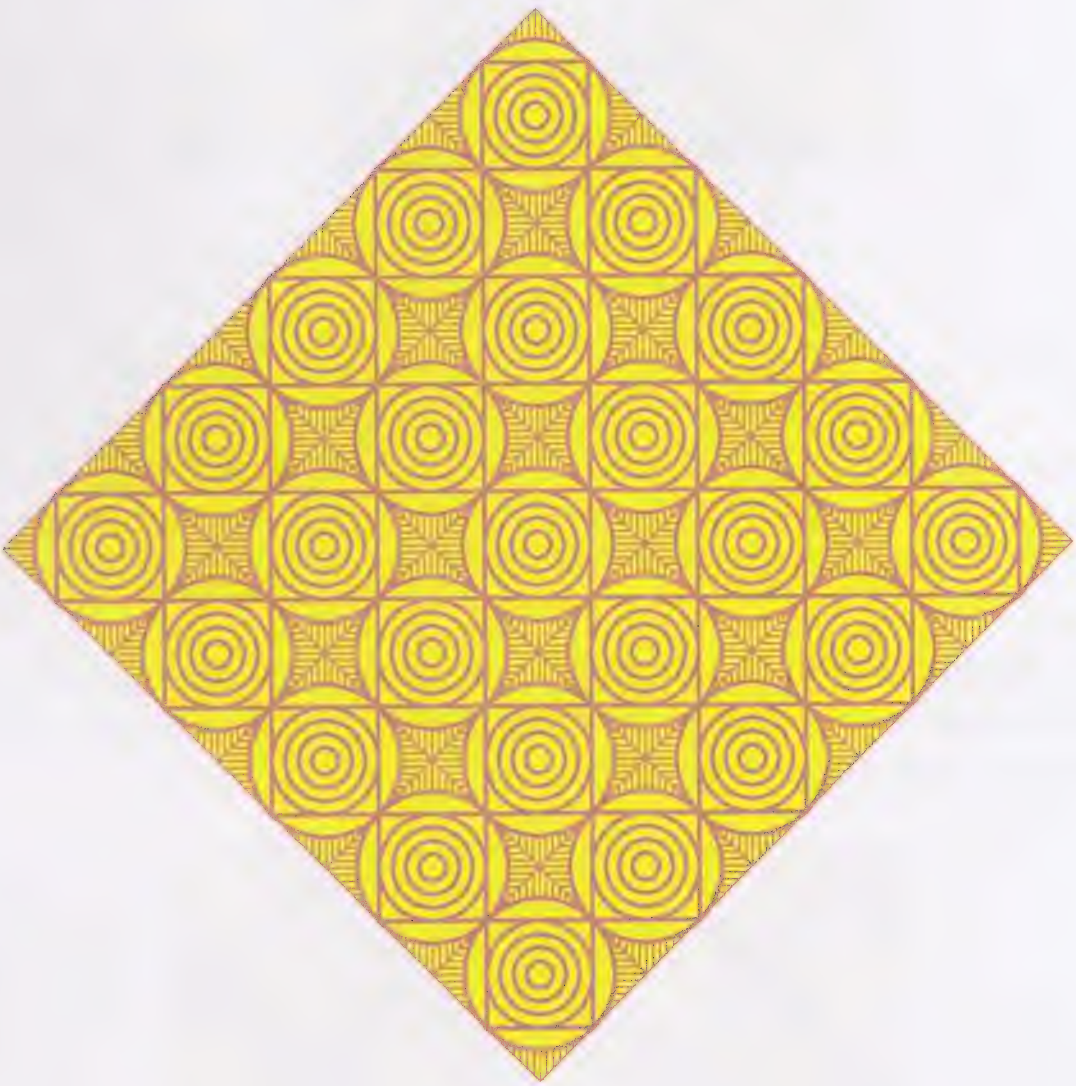


Figure 3.27

3.4 *'Chequer boards'*

Figure 3.27 shows a beautiful Angolan motif, used for the decoration of wooden sculpture. The ground pattern of this design is

displayed in Figure 3.28. It is composed of two superimposed square grids. Each square of the oblique grid is composed of a small square of the second grid, surrounded by four congruent triangles (see Figure 3.29). The area of these four triangles together is equal to that of a small square (see Figure 3.30). In other words, the area of an oblique square is twice the area of a square in the horizontal position.

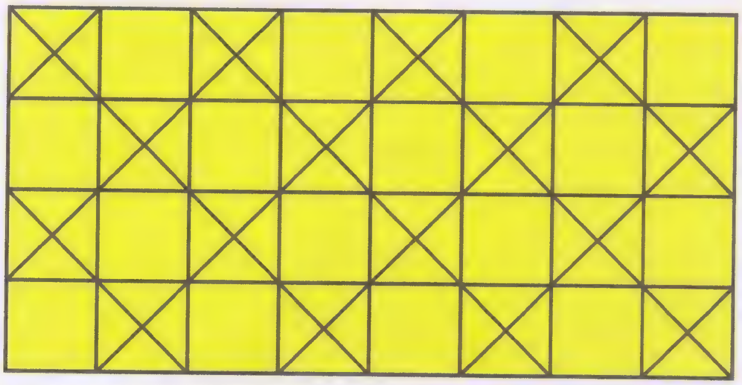


Figure 3.28

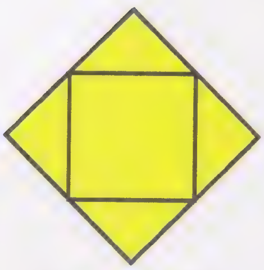


Figure 3.29

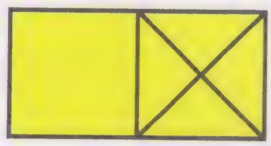


Figure 3.30

Another way to describe the ground pattern is to say that it is that of a chequer board, wherein the diagonals of the dark coloured squares are drawn (see Figure 3.31).

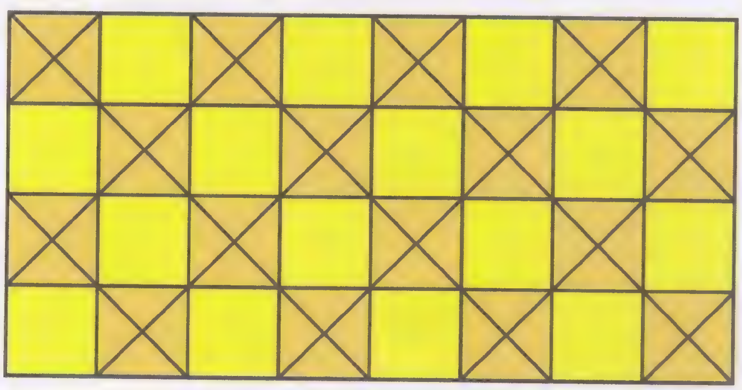


Figure 3.31

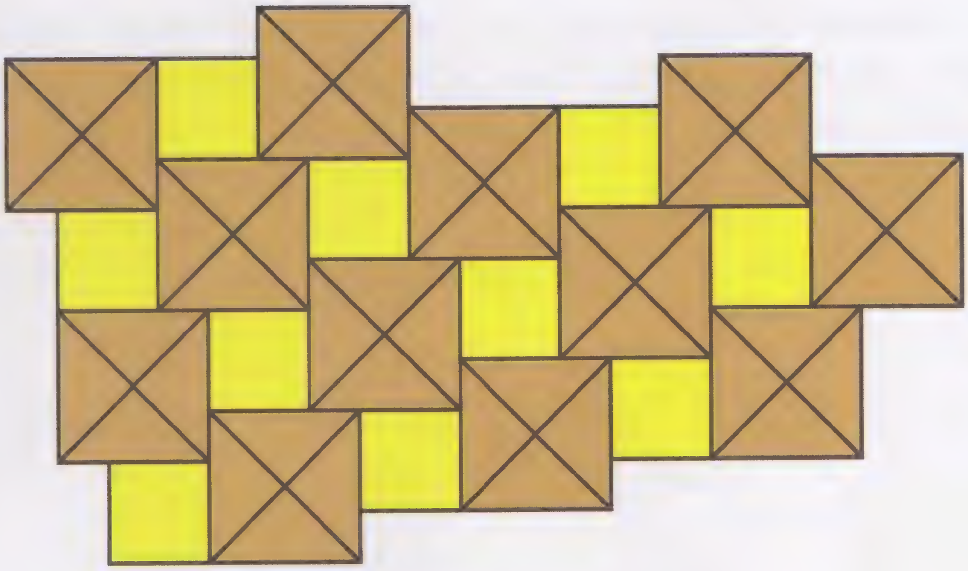


Figure 3.32

Now we may construct such a board with light and dark coloured chequers of different sizes (see Figure 3.32). Once more, each light coloured square is surrounded by four congruent triangles (Figure 3.33).

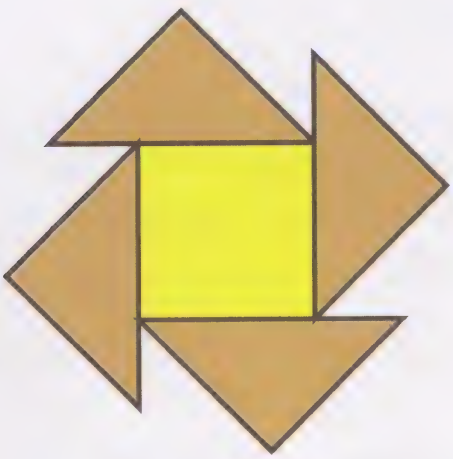


Figure 3.33

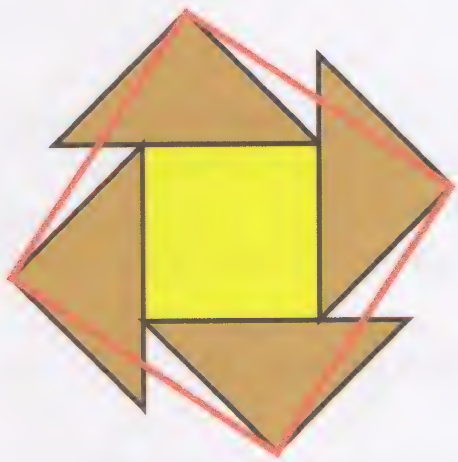


Figure 3.34

The centres of four neighbouring dark coloured chequers constitute the vertices of a new square (see Figure 3.34) that has an area (**C**) equal to the sum of the areas of a light coloured square (**A**) and four dark coloured triangles. As four dark coloured triangles together form a dark coloured chequer with area **B**, we find:

$$A + B = C.$$

Let ***a***, ***b*** and *c* denote the sides of these squares. In order to go from one vertex of a C-square to another (Figure 3.35), we may move

first a distance *b* horizontally to the right and then a distance *a* vertically upward. In other words, a triangle with sides *a*, *b* and *c* is right-angled, and

$$A + B = C$$

implies

$$a^2 + b^2 = c^2.$$

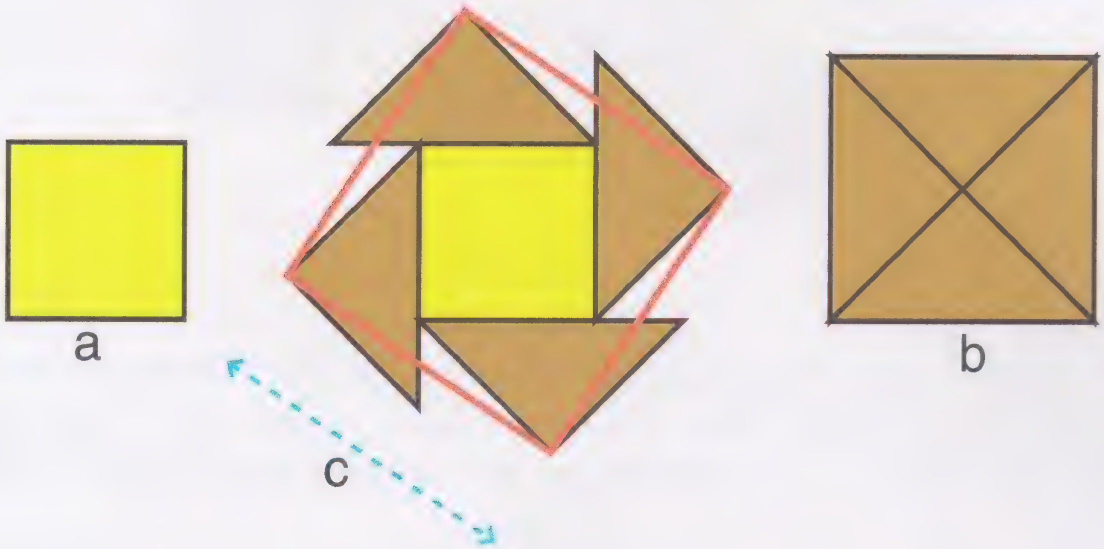


Figure 3.35

The last steps of this reasoning (from Figure 3.33 onwards) may be easily transformed into a proof, already known by Abu-l-Wafa (940-997 A.D.), a famous mathematician and astronomer who worked in Baghdad (Youschkevitch, 1976, p. 110).

3.5 *Two new proofs* *

Let us consider two arbitrary squares, both divided by their diagonals into four congruent triangles (see Figure 3.36). Let *a* and *b* be the lengths of the sides of the squares.

Will it be possible to construct a design with fourfold symmetry, composed of the eight triangles into which the two squares have been dissected? (see the examples in Figure 3.37).

Figure 3.38 displays another design with fourfold symmetry and made up of the 8 triangles. In its middle an empty square hole appears.

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* Found by the author in 1991.

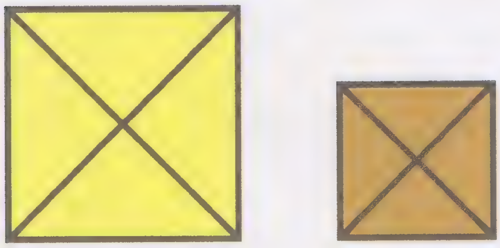


Figure 3.36

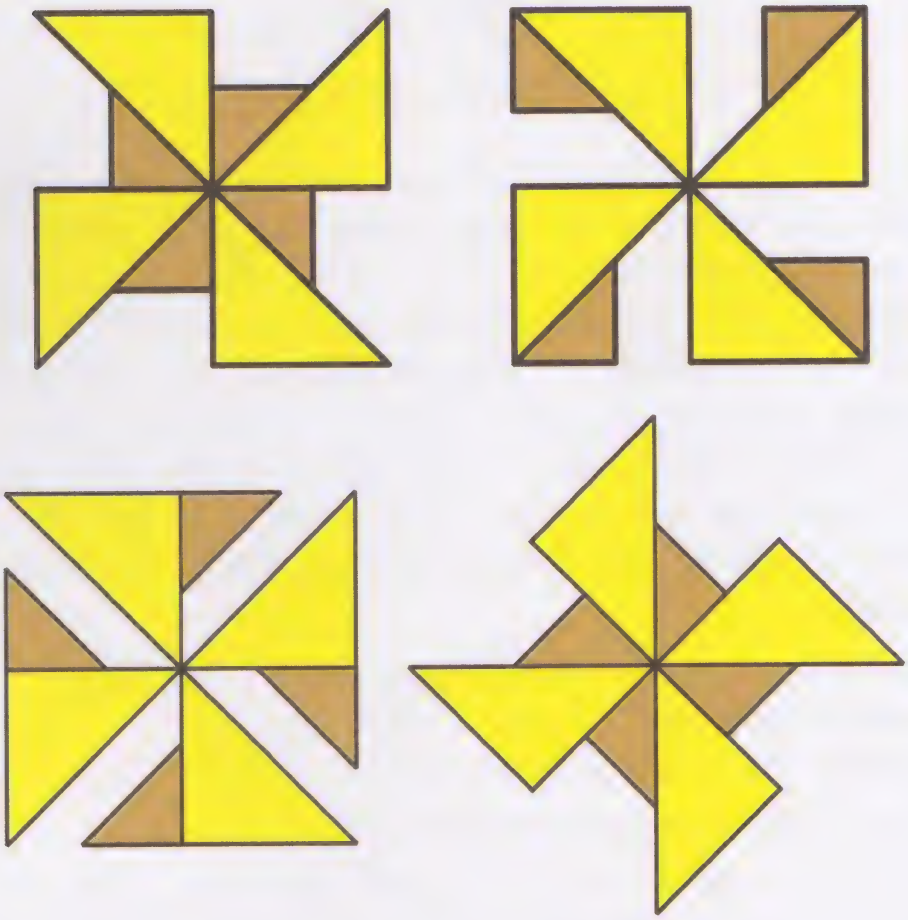


Figure 3.37

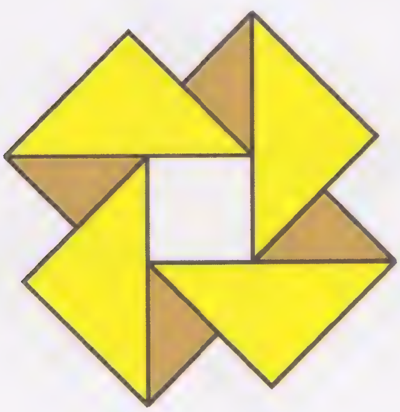


Figure 3.38

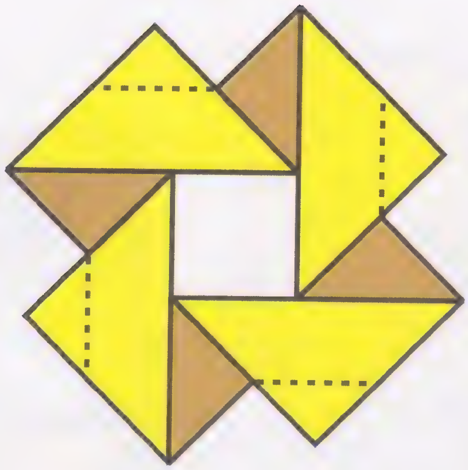


Figure 3.39

When we cuts off a small triangle, as illustrated in Figure 3.39, from each of the triangles of the larger initial square, then we may fill the empty square with them (see Figure 3.40). In this way we obtain a dodecagon (12-gon), whose area is equal to the area of the square that we obtain when we join those vertices of the 12-gon, where a triangle of the small initial square and a triangle of the larger initial square meet (see Figure 3.41). Let c be the length of the side of this new square. Taking the construction of the design into account, we find that a triangle with sides a, b and c is right-angled (see Figure 3.42). To summarize:

sum of the areas of the initial squares = $a^2 + b^2$ =
= area of the design (Figure 3.38) =
= area of the 12-gon (Figure 3.40) =
= area of the new square (Figure 3.41) = c^2.

Therefore: $a^2 + b^2 = c^2$, that is, we arrived at the Theorem of Pythagoras. It is not difficult to justify the steps of the proof that has been sketched.

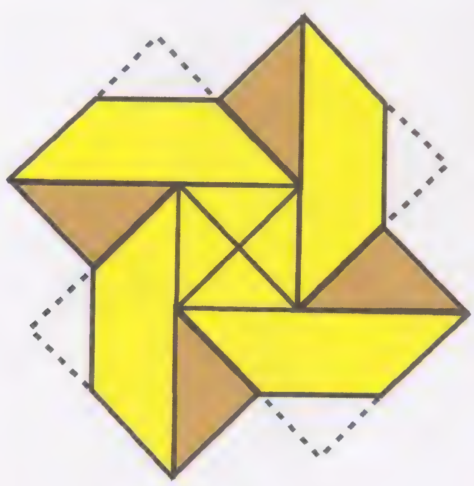


Figure 3.40

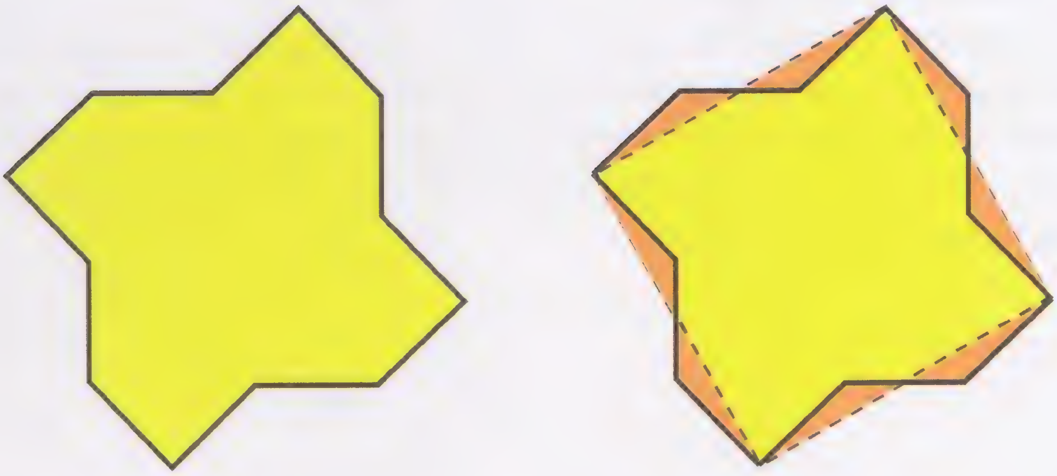


Figure 3.41

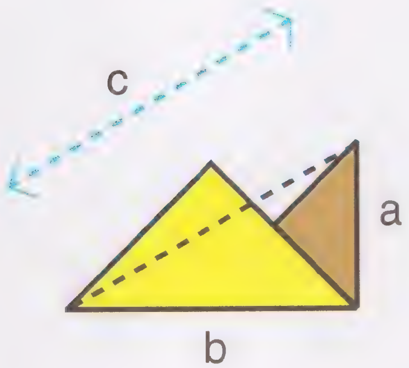


Figure 3.42

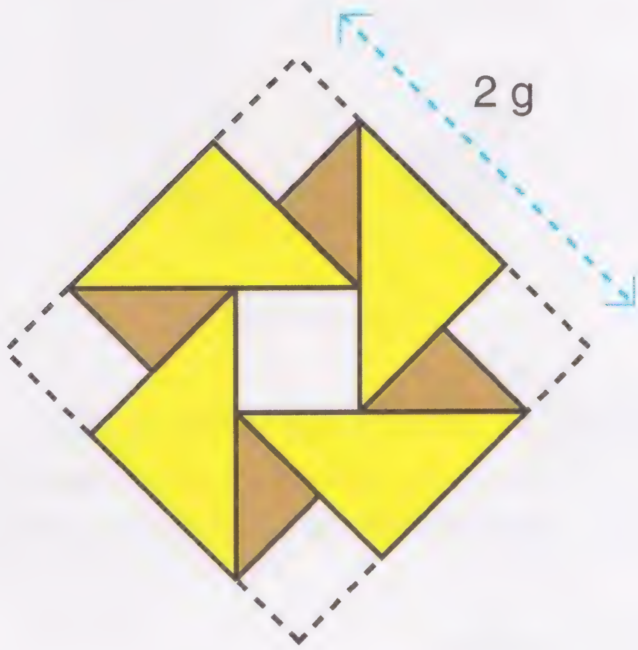


Figure 3.43

Returning to Figure 3.38, we may find a more algebraic proof too. Let us construct the circumscribed square of the design (Figure 3.43) and the square of side *c* (see Figure 3.44). Let *f* and *g* be the lengths of the sides of the triangles of the squares, whose hypotenuses measure *a* and *b* (see Figure 3.45). In this way the circumscribed square has a side of length *2g*.

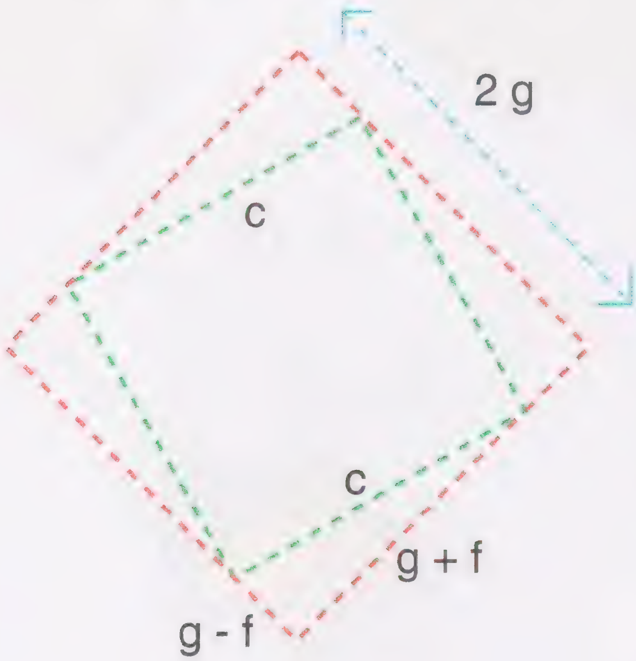


Figure 3.44

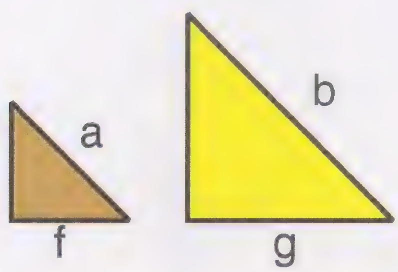


Figure 3.45

The area $[(2g)^2]$ of the circumscribed square is equal to the area (c^2) of the square of side *c* together with the areas of the four triangles in the corners. The right-angled triangles in the corners have **g+f** and **g-f** as sides.

In this way we see that:

$$(2g)^2 = c^2 + 4.\tfrac{1}{2}(g+f)(g-f)\cdot,$$

$$4g^2 = c^2 + 2g^2 - 2f^2\ ,$$

$$2f^2 + 2g^2 = c^2.$$

Taking into account that $2f^2 = a^2$ and $2g^2 = b^2$ (see Figure 3.46), we arrive at the final conclusion that

$$a^2 + b^2 = c^2,$$

that is, we arrive at the Theorem of Pythagoras.

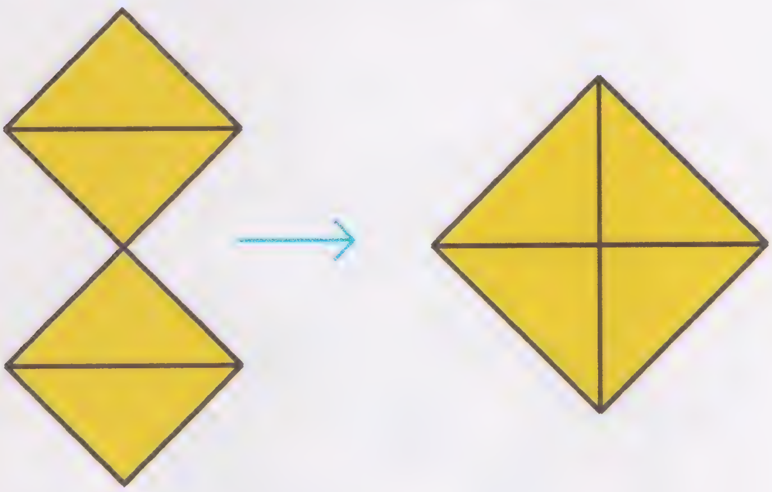


Figure 3.46

Cover of the first edition in Portuguese (1992)

Chapter 4

'PITÁGORAS', SIMILAR TRIANGLES AND THE "ELEPHANTS'-DEFENCE" DESIGN OF THE KUBA (CENTRAL AFRICA) *

The (Ba)Kuba people inhabit the central part of the Congo basin (in today's Democratic Republic of Congo), living in the savannah south of the dense equatorial forest. The Kuba had constituted a strong and secular kingdom. Their metallic products, like weapons and jewellery, are famous. The villages themselves had specialized in certain types of craft, like the production of ornamented wooden boxes and cups, velvet carpets, copper pipes, raffia cloths, etc.

The beautiful decorative art of the Kuba (see the examples in Figure 4.1) attracted not only the attention of artists from all over the world (Meurant, 1987), but also that of mathematicians. Donald Crowe (USA) analysed symmetry patterns occurring in Kuba designs and found their mathematical variety and richness evident (Crowe, 1971; Zaslavsky, 1973, chap. 14).

In this chapter I shall present some Kuba patterns and show how these designs may serve as an interesting starting point in the study of geometry.

---

* Example given during the invited lecture 'About the multiculturalisation of mathematics education' (1st Ibero-American Congress on Mathematics Education, Seville, Spain, September 23-29, 1990).

Figure 4.1

A particular case of the Pythagorean proposition

Kuba designs, like the one illustrated in Figure 4.2 may serve to discover the Pythagorean proposition in the particular case where both sides are equal (see Figure 4.3).

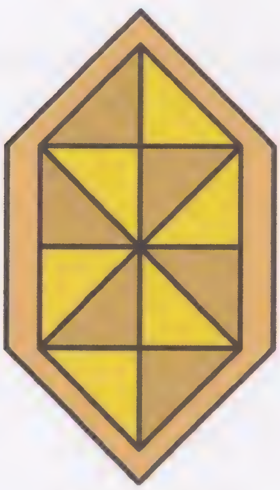


Figure 4.2

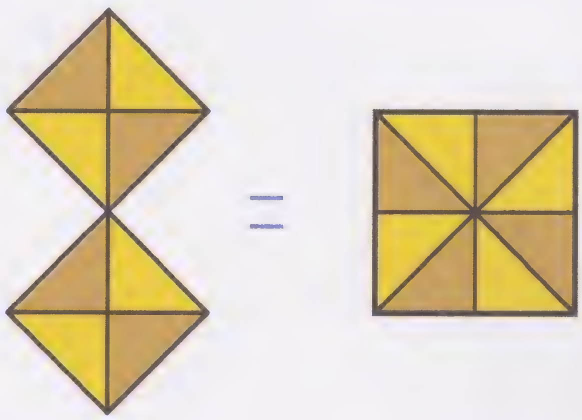


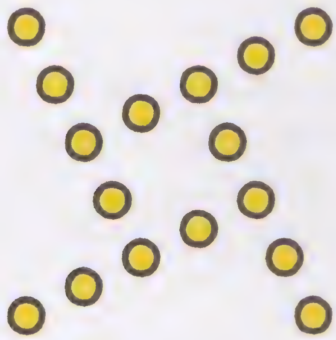
Figure 4.3

The MONGO and MWOONG designs

Figure 4.4 shows two tattooing motifs formerly used by the Ngongo, one of the ethnic groups that belonged to the Kuba kingdom. Both display a fourfold rotational symmetry (rotational symmetry of

order 4).

When we draw lines between the consecutive ends of these tattooing motifs, we obtain a design similar to the Kuba engravings illustrated in Figures 4.5 and 4.6.



Tattoos
Figure 4.4

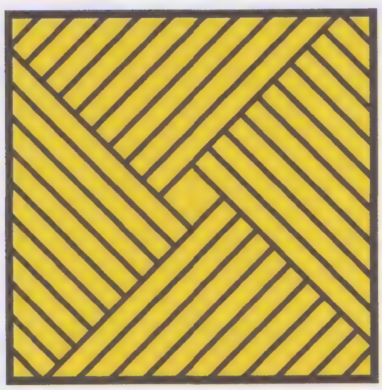
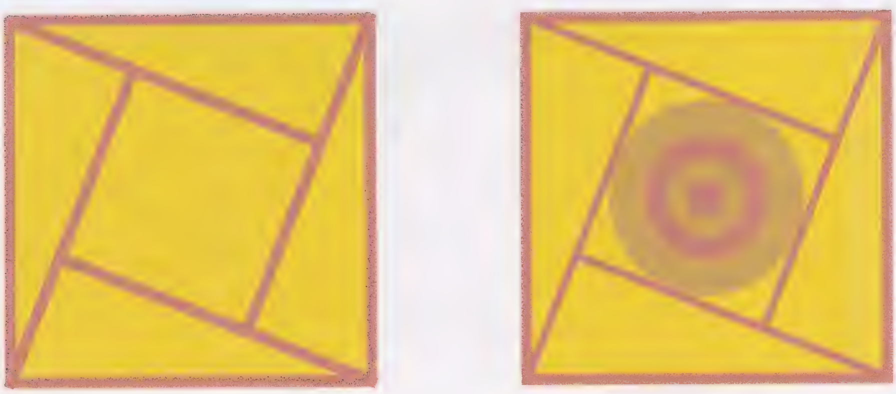


Figure 4.5



Kuba engravings
Figure 4.6

The Ngongo call the engraving motif in Figure 4.5 ***mongo***, that

is, knee. [1]. The (Bu)Shongo - the dominant group in the old Kuba kingdom – call the engraving pattern in Figure 4.6a ***mwoong***, that is, "elephants' defence," and the motif in Figure 4.6b ***ikwaakl'imwoong***, that is, "deformed elephants' defence" (Meurant, 1987, p. 177).

To arrive at the Pythagorean proposition

The Kuba engraving design in Figure 4.6a is well known from the history of Mathematics in Asia (China, India). When we place several copies of the design together, we may discover, 'reinvent' and even prove, easily and geometrically, the so-called Theorem of Pythagoras, as Figure 4.7 illustrates.

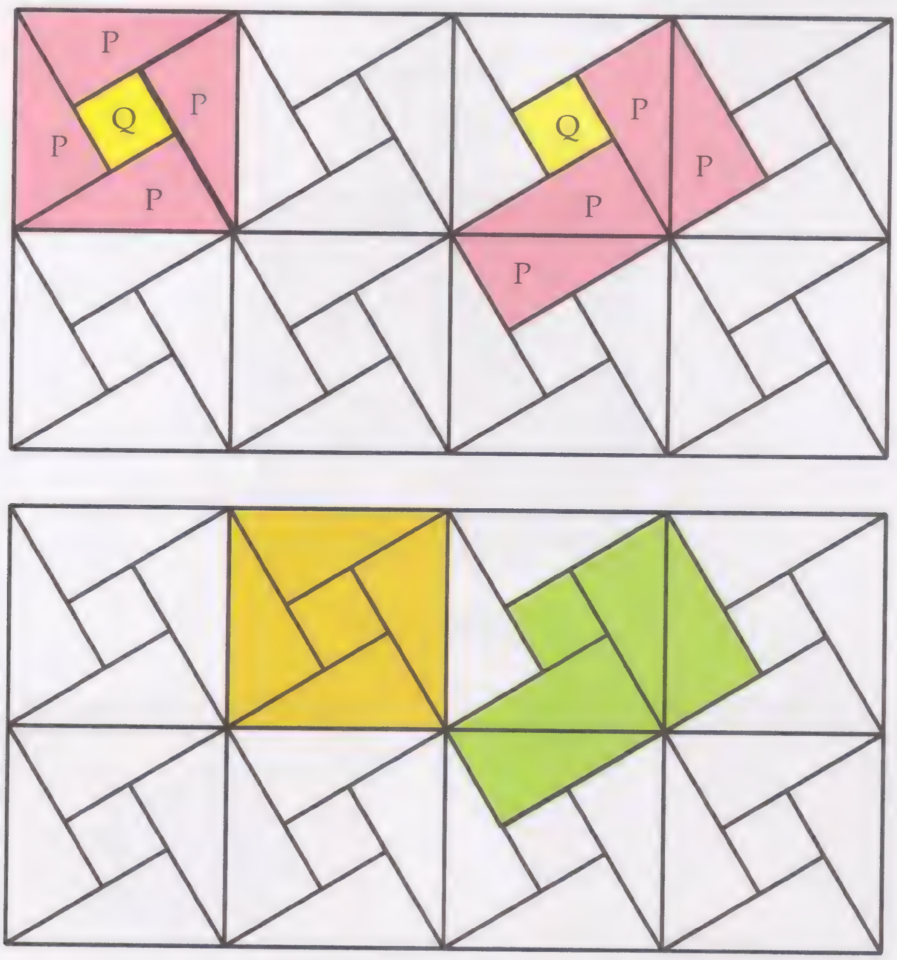


Figure 4.7

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1 The motifs presented in Figures 4.4 and 4.5 are related as they have in the Ngongo tradition the same name, ***mongo***, as the weaving pattern illustrated in Figure 4.21. The tattoo motif has its origin in basketry, having been abstracted, that is, isolated, from its original context.

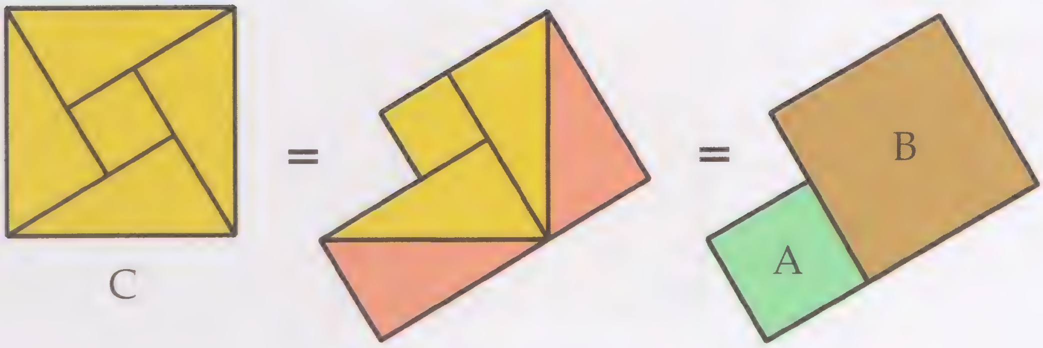


Figure 4.7 (conclusion)

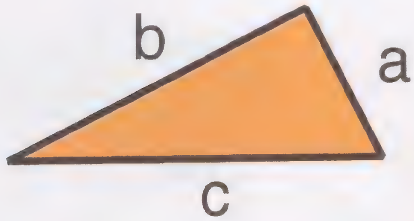


Figure 4.8

On the other hand, knowing the identity $(b-a)^2 = b^2 - 2ba + a^2$. and the formulas for the determination of the areas of squares and right-angled triangles, we may arrive, in the following algebraic-geometric way, at the same conclusion. The area of the small central square is equal to $(b-a)^2$ and the combined areas of the four neighbouring right-angled triangles are 2ab. Therefore we have:

(1) $$c^2 = (b-a)^2 + 2ab = b^2 + a^2.$$

An ornamental variant and a generalization of the Pythagorean proposition

Figure 4.9 shows a Kuba variant of the elephants' defence design.
[2]

---

2 During the 6th International Congress on Mathematical Education (August 1988), the author had the opportunity to visit the permanent exhibition 'From clans to civilizations' in the Ethnographic Museum of Budapest (Hungary). Room VIII displays art and craft of the Kuba. The elephants' defence design may be seen on several objects. This variant in Figure 4.9 is carved on the wall and cover of a beautiful wooden box.

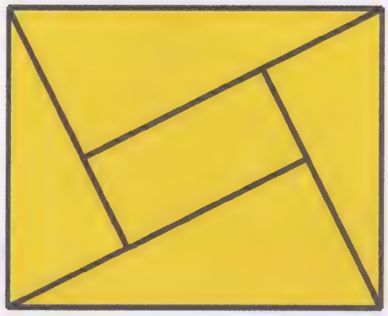


Figure 4.9

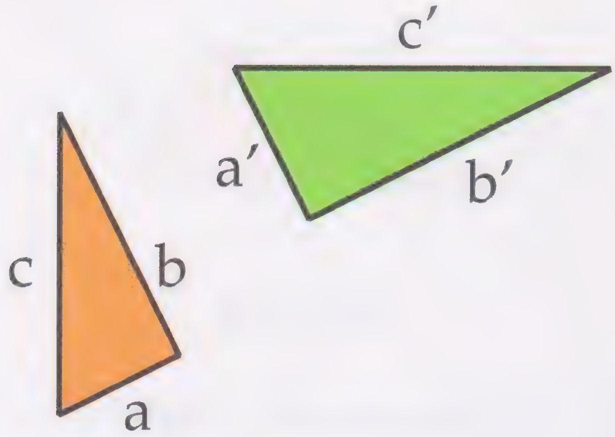


Figure 4.10

The big rectangle is composed of two pairs of similar triangles (see Figure 4.10) and a little rectangle at the centre.

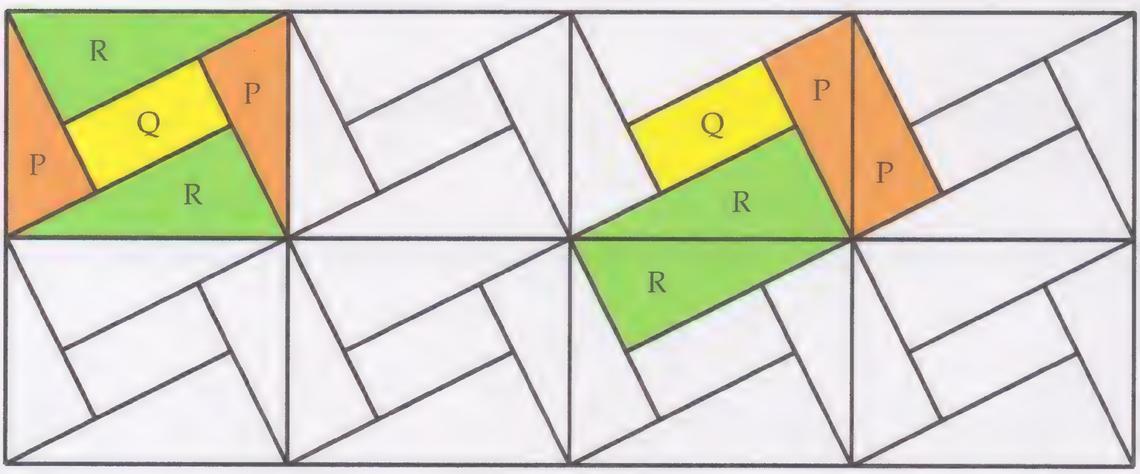


Figure 4.11

When we place several copies of this variant together (see Figure 4.11), we may discover/invent and prove the following generalization of the Pythagorean proposition:

(2) $c\,c' = a\,a' + b\,b'$ (see Figure 4.12)

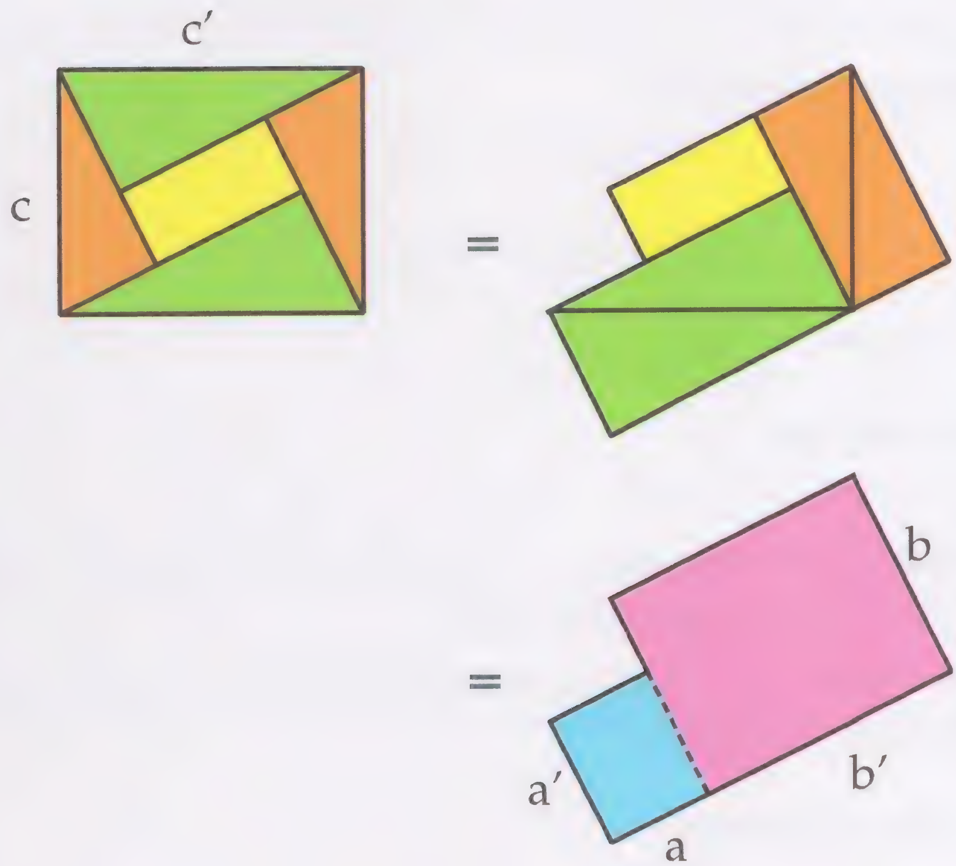


Figure 4.12

On the other hand, it is possible to arrive at the same conclusion by algebraic-geometrical reasoning. The area of the central rectangle is equal to $(b' - a)(b - a')$ and the areas of the four neighbouring right-angled triangles are together $ab + a'b'$. Therefore

$$c\,c' = (b' - a)\,(b - a') + a\,b + a'b' = a\,a' + b\,b'.$$

A combination of Pythagorean Proposition with its generalization

What result may be obtained when we combine the Pythagorean theorem with it generalization (2) ?

(2) $c\,c' = a\,a' + b\,b'$

Combining (2) with the Theorem of Pythagoras

(1) $c^2 = a^2 + b^2$

and

(1') $$(c')^2 = (a')^2 + (b')^2,$$
we obtain

(3) $$(a\,a' + b\,b')^2 = (c\,c')^2 = c^2\,(c')^2 = [a^2 + b^2][(a')^2 + (b')^2].$$

This leads immediately to

(4) $$2\,a\,a'\,b\,b' = a^2\,(b')^2 + b^2\,(a')^2$$
and

(5) $$(a\,b' - b\,a')^2 = 0.$$

In this way, we see that

(6) $$a\,b' = b\,a',$$
which implies that

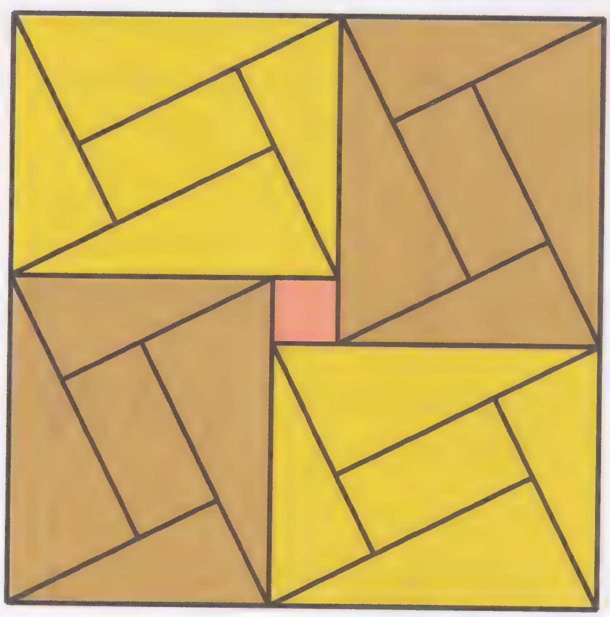
(7) $$a : b = a' : b'.$$

In other words, the ratios of the sides (taken in the same order) of similar right-angled triangles are equal.

The deduction presented here is algebraic. Does a (purely) geometrical alternative not exist?

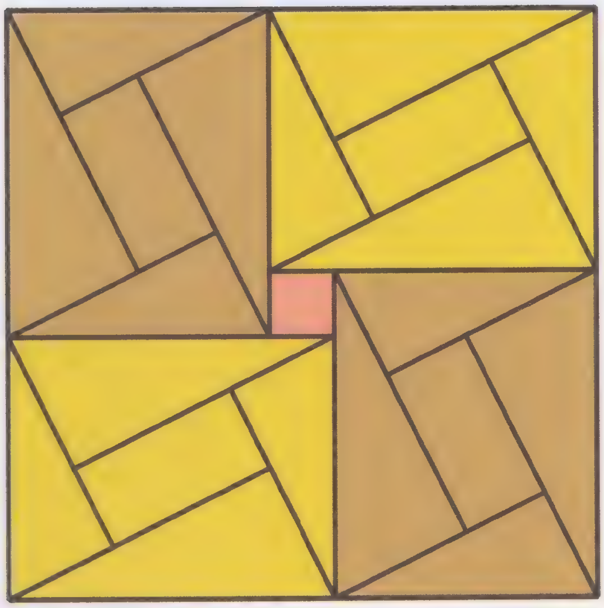
A geometrical alternative

Let us return to the rectangular Kuba variant of the elephants' defence pattern (Figure 4.9). How can we join some of these ornamental rectangles in order to obtain a square?



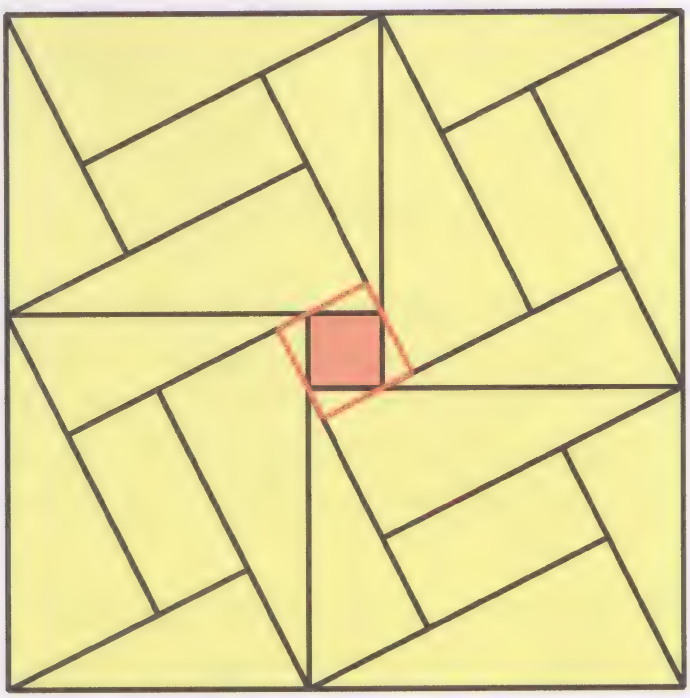
a

Figure 4.13

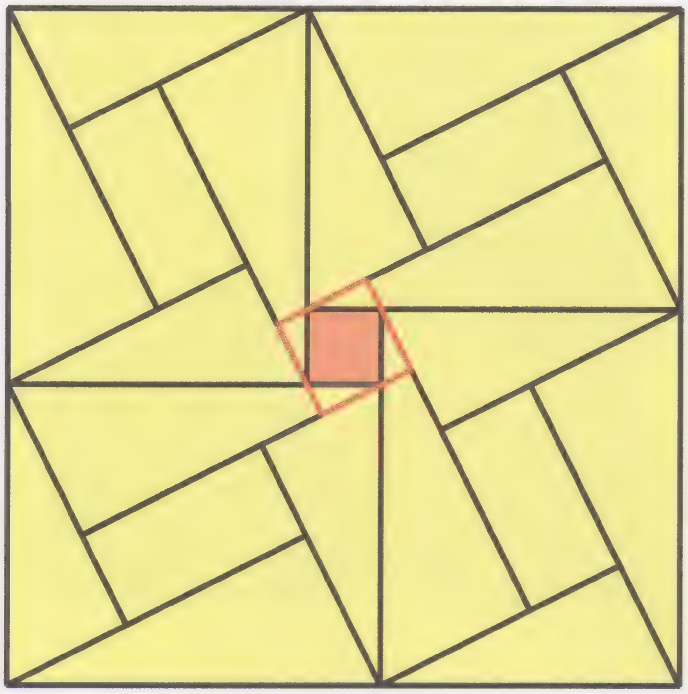


b
Figure 4.13

Figures 4.13a and b show two possibilities of obtaining a square of side c + c'. In both cases, a 'hole' appears in the centre, whose sides measure c'-c. When we extend the sides that end at the vertices of the square 'hole' until they encounter other sides, we obtain Figures 4.14a and b.



a
Figure 4.14



b
Figure 4.14 (conclusion)

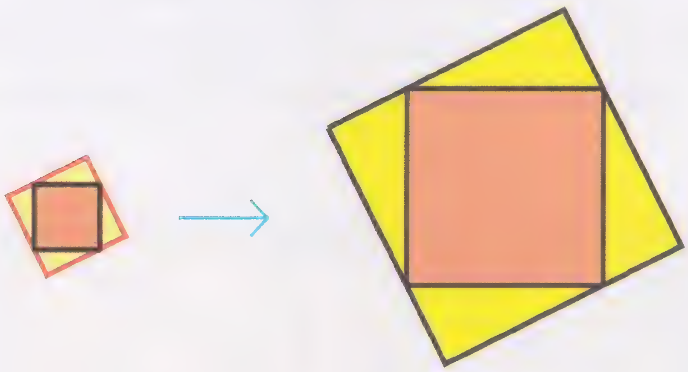
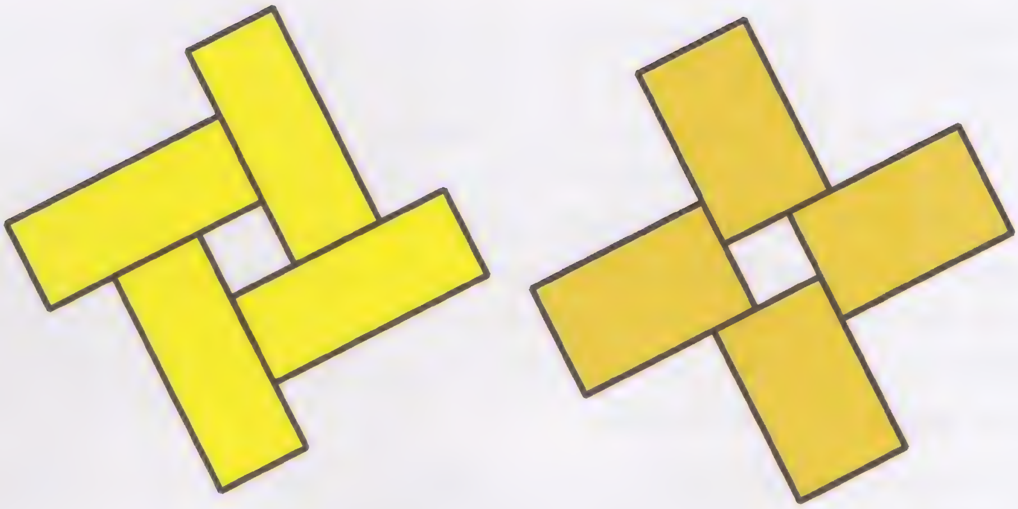


Figure 4.15

In both cases, four right-angled triangles appear around the square 'hole' (see Figure 4.15), which, together with the square 'hole', form a new square of side (a'+b')-(a+b). In both cases, around this new central square appear four rectangles which constitute designs that are similar (see Figure 4.16) to the Kuba tattooing displayed in Figure 4.4b.

As the large squares are congruent and the small squares are also congruent, we arrive at the conclusion that the designs themselves (see Figure 4.16) are equal in area. Both designs are composed of four rectangles. Therefore, the area (**ab'**) of one of these rectangles of the first design (Figure 4.16a) is equal to the area (**a'b**) of one of the rectangles of the second design (Figure 4.16b):

(8) $ab' = a'b.$



a b

Figure 4.16

In other words,

(9) $a : b = a' : b',$

that is, we have arrived geometrically at the conclusion that the ratios of the sides (taken in the same order) of similar right-angled triangles are equal.

It turns out to be easy to prove the Fundamental Theorem of Similar Triangles on the basis of the foregoing result. Also we may introduce, without any difficulty, the concepts of tangent, sine and cosine of an acute angle. Application of theorem (2) in the case of the triangles in Figure 4.17 leads to the trigonometric formula

(10) $c = a \sin \alpha + b \sin \beta.$

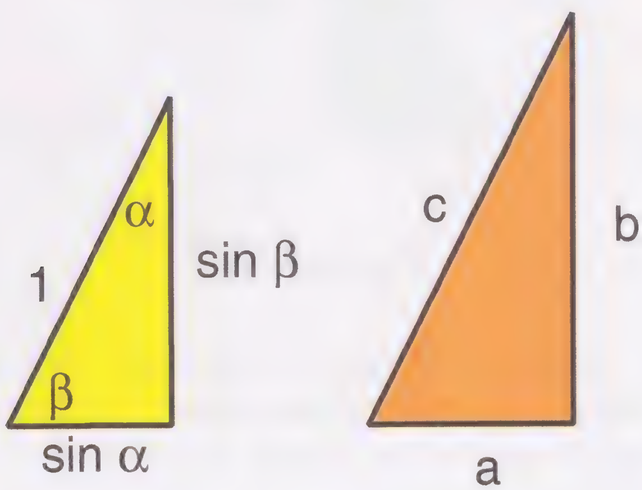


Figure 4.17

Other geometrical alternatives

Inspired by the Kuba designs, we saw that there exists a (purely) geometrical deduction of the theorem that says that the ratios of the sides (taken in the same order) of similar right-angled triangles are equal. As in the schoolbooks we know this is either not proven or is proven on the basis of more general knowledge (Fundamental Theorem of Similar Triangles), I was led to look for other alternatives. I found the following: both alternatives may easily be used in the mathematics classroom, for instance, when the trigonometric ratios of an acute angle are introduced.

Let us consider a right-angled triangle with sides a+a', b+b' and c+c' (Figure 4.18a). The right-angled triangles of sides *a*, *b*, *c* and *a'*, *b'*, *c'* may fit into it in various ways. Figures 4.18b and c present two possibilities. The rectangles that remain have obviously the same areas. Therefore:

(8) $$ab' = a'b.$$

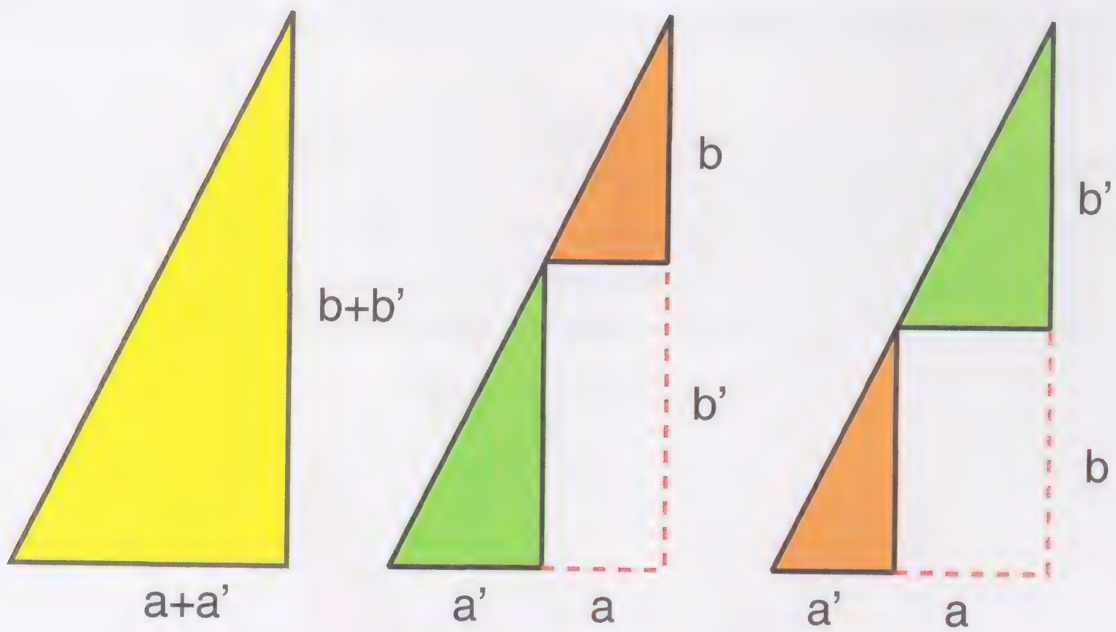


Figure 4.18

Let us now consider right-angled triangles with sides *a*, *b*, *c* and *a'*, *b'*, *c'*. We may join them as illustrated in Figures 4.20a and b. In both cases, we may consider the figure so obtained as composed of a rectangle and a right-angled triangle with sides a'-a, b'-b and c'-c (see Figure 4.20). Therefore, both rectangles have the same area, and we

may conclude:

(8) $ab' = a'b.$

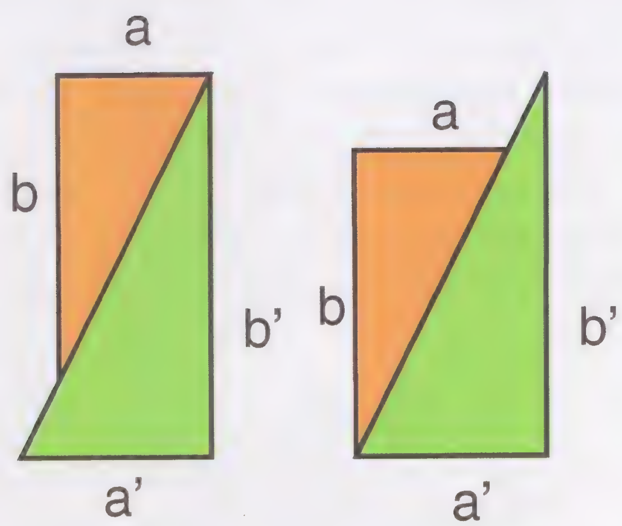


Figure 4.19

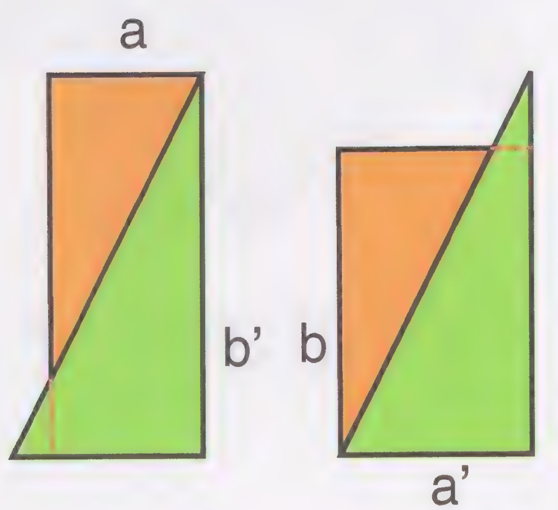


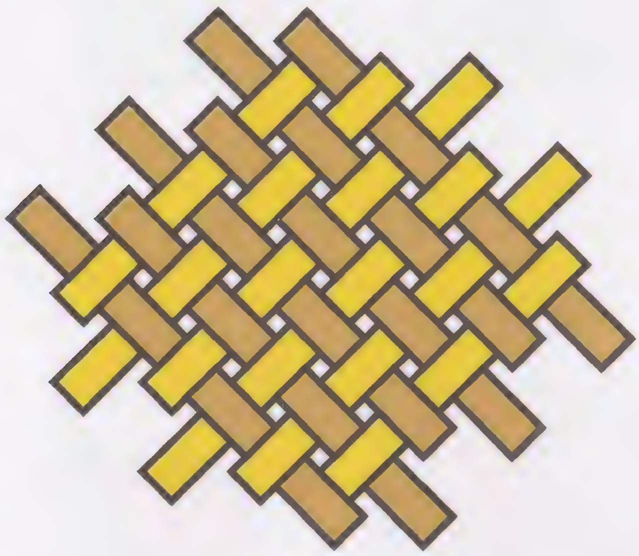
Figure 4.20

'Pythagoras' once more

The well-known drawing in Figure 4.17 constitutes an invitation for the discovery of some more proofs of the Theorem of Pythagoras.

Final considerations

Kuba designs may serve as an attractive starting point in the mathematics classroom for the (re)discovery / invention and proof of the Theorem of Pythagoras, of one of its generalizations and of (a particular case of) the Fundamental Theorem of Similar Triangles. It was also shown how the considered Kuba designs may be used for the introduction of trigonometric ratios. Our reflection about the possibilities of incorporating cultural elements of the Kuba into the teaching of geometry inspired us to find some didactical alternatives.



Over-one-under-one weaving
Figure 4.21

Chapter 5

A WIDESPREAD DECORATIVE MOTIF AND THE THEOREM OF PYTHAGORAS *

A widespread decorative motif

Figure 5.1

Figure 5.1 displays variations of a decorative motif with a long tradition all over Africa and the world. It was already known in Ancient Egypt. It appears on a painted basket in the tomb of Rekhmire (Thebes, 18th dynasty, c. 1475-1420 B.C.) (see Figure 5.2). Other examples from distinct parts from Africa are shown in Figure 5.3. Among the Cokwe (Angola) the pattern is known as ***manda a mbaci***, that is, tortoise-shield (Bastin, 1961, p. 114, 116). The Ovimbunda (Angola) call it ***olombungulu*** (star), ***ononguinguinini*** (ants), or ***alende*** (clouds) (Hauenstein, 1988, p. 39, 50, 54). It appears as a motif on beaded combs from the Yao (Mozambique, Malawi) (Carey, 1986, p. 29). It is frequently seen on baskets from all over the continent.

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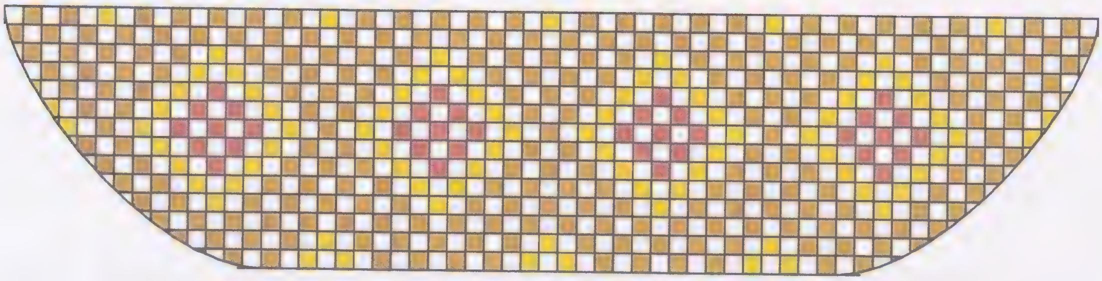
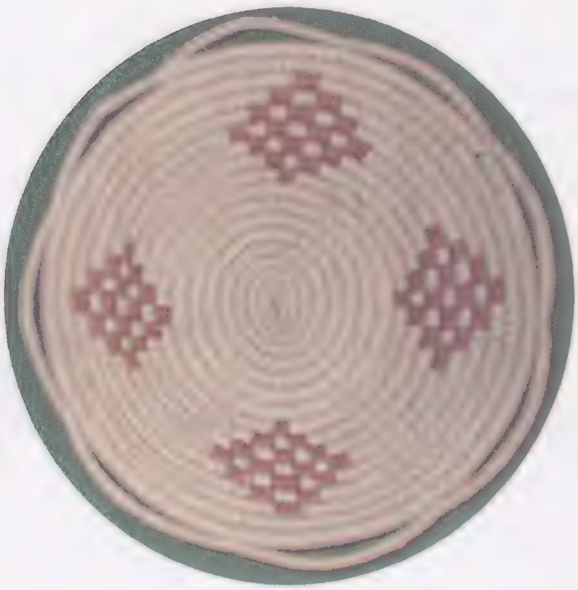
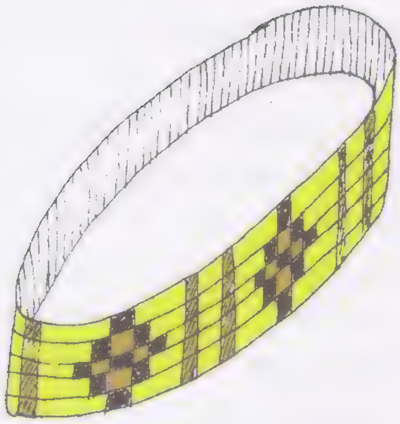
* Adapted from a paper published in the international journal *For the Learning of Mathematics* (Montreal, Vol. 8, No. 1, 1988, 35-39).



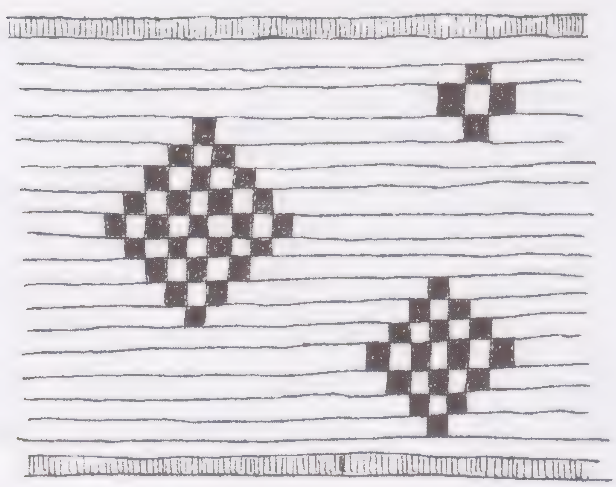
Basket from Ancient Egypt
Figure 5.2



Basket from Lesotho
Figure 5.3



Bracelet from Senegal
a



Alesu motif from Angola
b

Figure 5.4

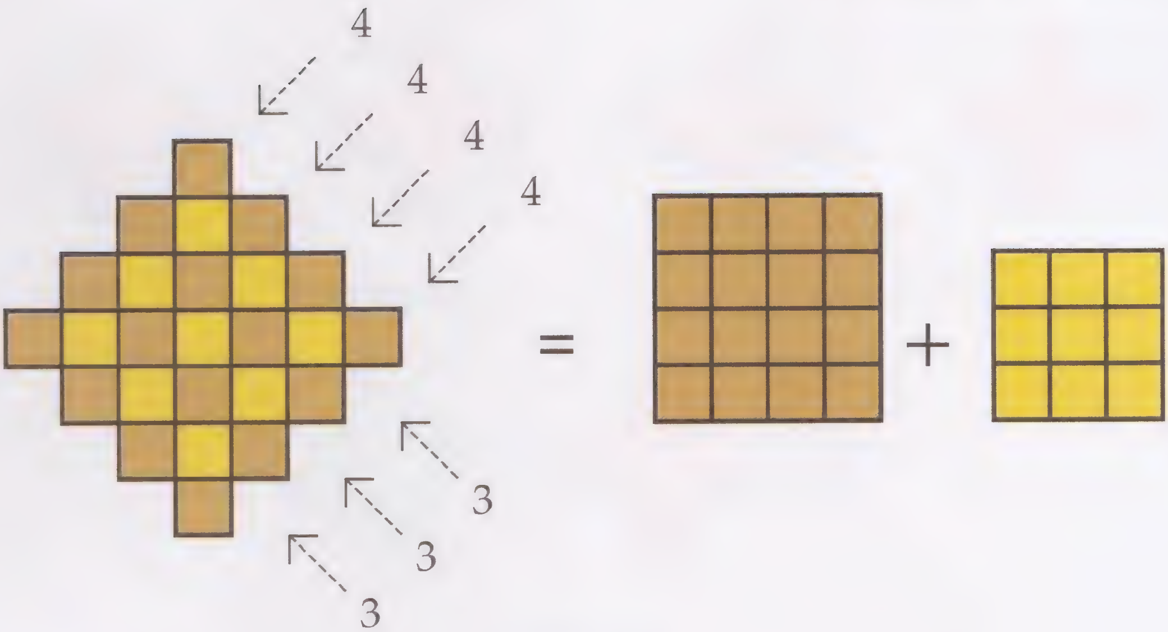


Figure 5.5

Discovering the Theorem of Pythagoras

Looking at the number of unit squares on each row of the 'star' in Figure 5.5, it is easy to see that the area of a 'star' is equal to the sum of areas of the 4x4 darker coloured square and of the 3x3 lighter coloured square. [1]

A 'star' (Figure 5.1) may also be called a ***toothed square***. A 'toothed square', especially one with many teeth, looks almost like a real square. So naturally the following question arises: is it possible to transform a toothed square into a real square of the same area? By experimentation (see Figure 5.6), the pupils may be led to draw the conclusion that this is indeed possible.

---

1 Students can also be led in many other ways to draw the conclusion that a 'toothed square' is equal in area to the sum of two real squares. For example, the teacher may ask them to transform a 'star' made of loose tiles of two colours into two monochromatic similar figures. Or, (s)he can ask them to cut off the biggest possible square from a 'star' made of paper or cardboard and to analyse which figures can be laid down with the other pieces (see Figure 5.14).

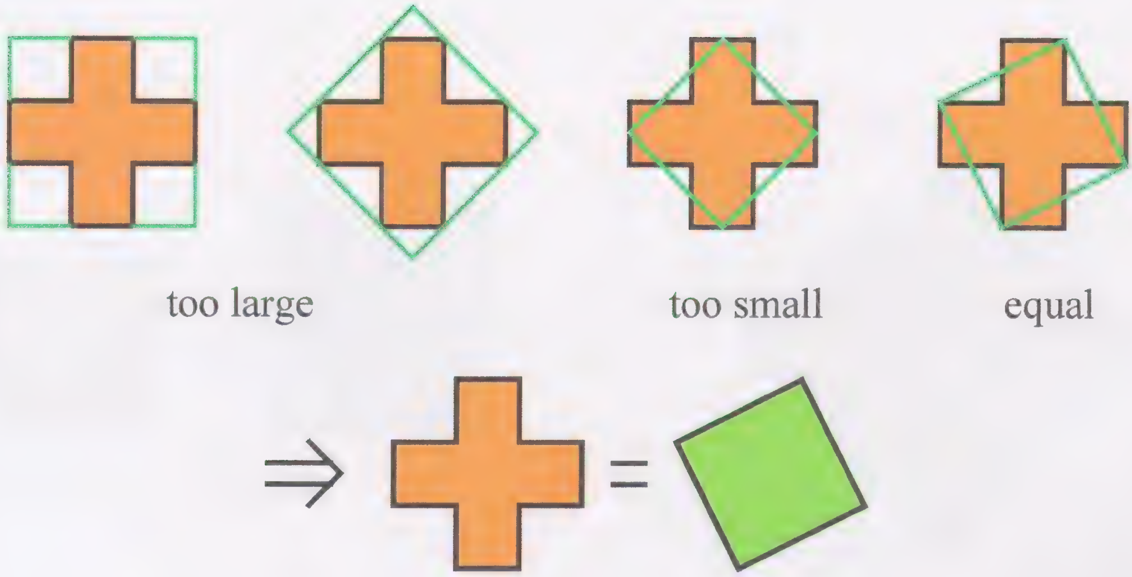
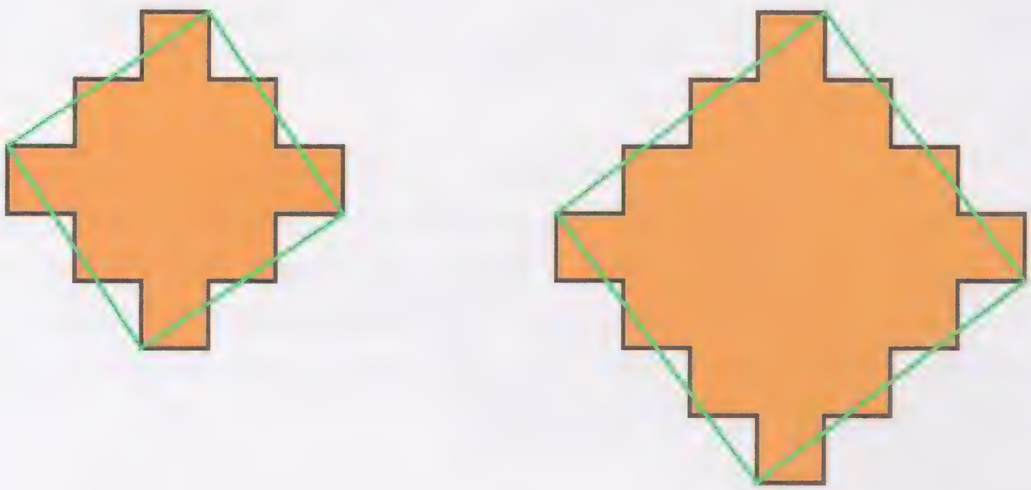


Figure 5.6

In Figure 5.5, we have seen that the area of a toothed square (T) is equal to the sum of the areas of two smaller squares (A and B):

T = A + B.



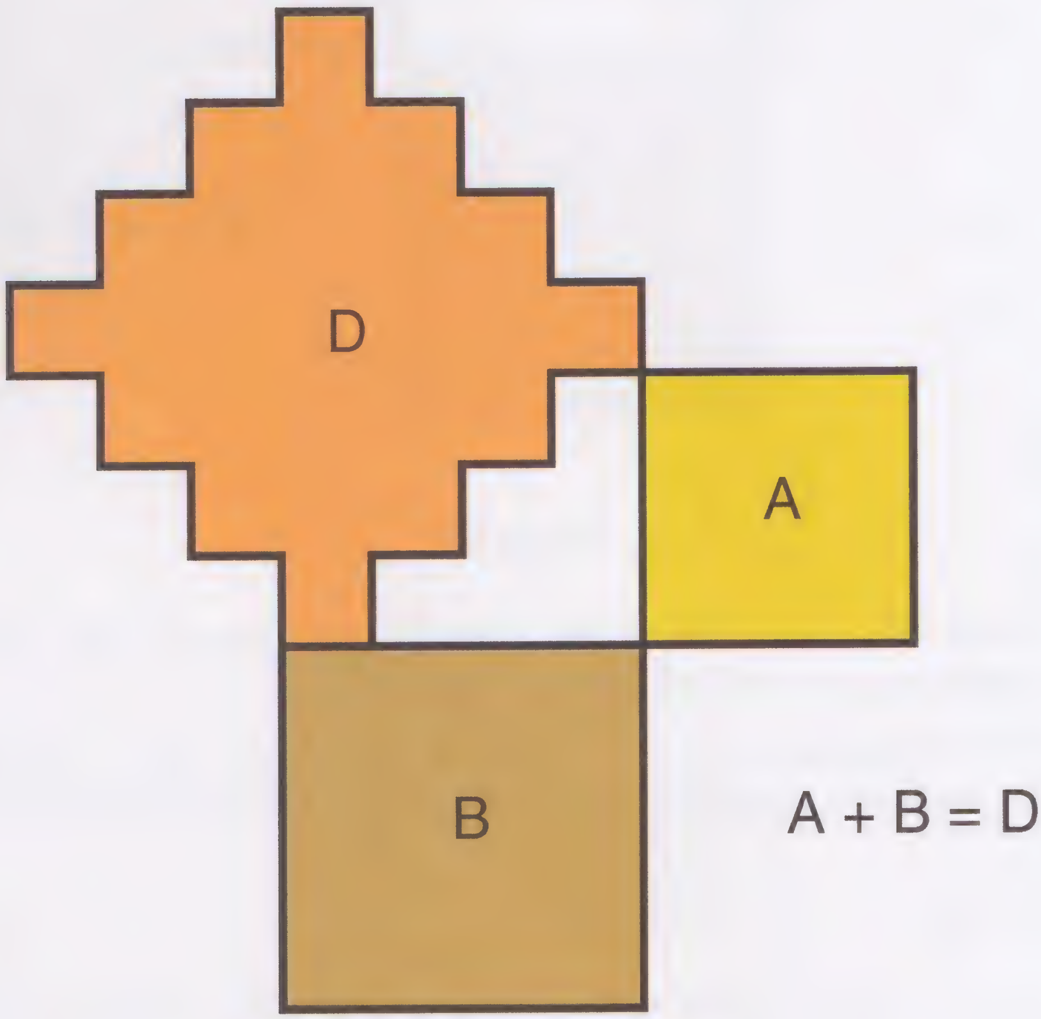
T = C
Figure 5.7

From Figure 5.7 we concluded that the area of a toothed square (T) is equal to the area of a 'real' square (C). Since C = T, we may conclude that

A + B = C.

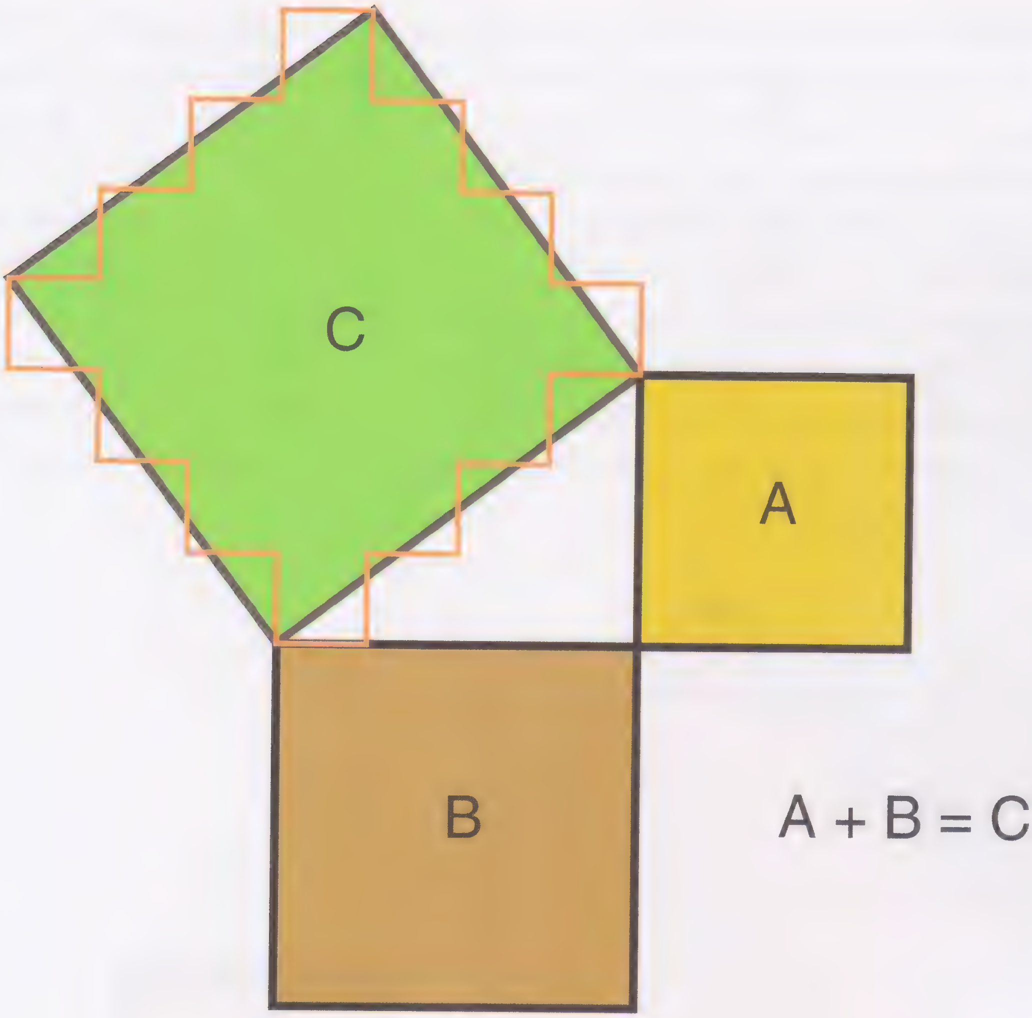
Do other relationships between these three squares exist? What does happen if we draw the toothed square and the two real squares

(into which it is decomposed) together on square grid paper, in such a way that they become 'neighbours'? Figure 5.8a shows a possible solution. When we now draw the last real square (area C) in the same figure, we arrive at the Pythagorean Theorem for the case of (*a*, *b*, *c*) right-angled triangles with *a:b = n:(n+1)*, where the initial toothed square has *n+1* teeth on each side. Figure 5.8b illustrates the Pythagorean Proposition for the special case of the (3, 4, 5) right-angled triangle. On the basis of these experiences, the pupils may be led to ***conjecture*** the Pythagorean Theorem in general. In this manner, toothed squares assume a ***heuristic value*** for the discovery of this important proposition.



a

Figure 5.8



b
Figure 5.8

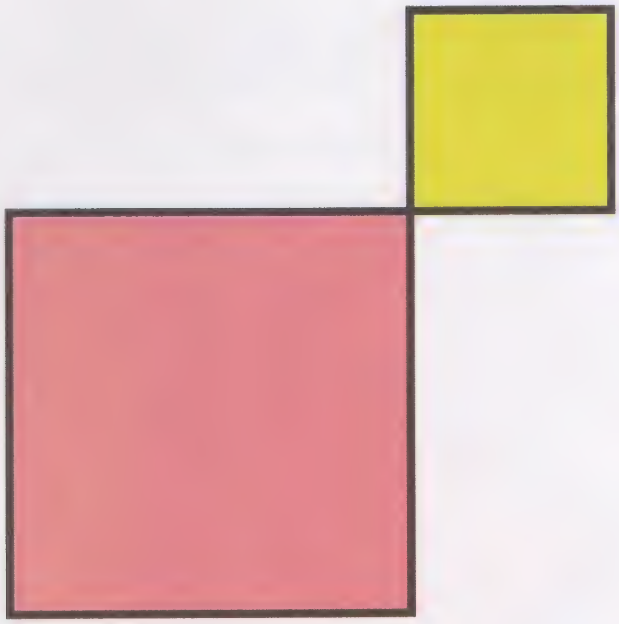
Does the same discovery process also suggest any (new) ***demonstrations*** for the Pythagorean Theorem?

What happens when we reverse the process? When we begin with two arbitrary squares and use them to generate a toothed square?

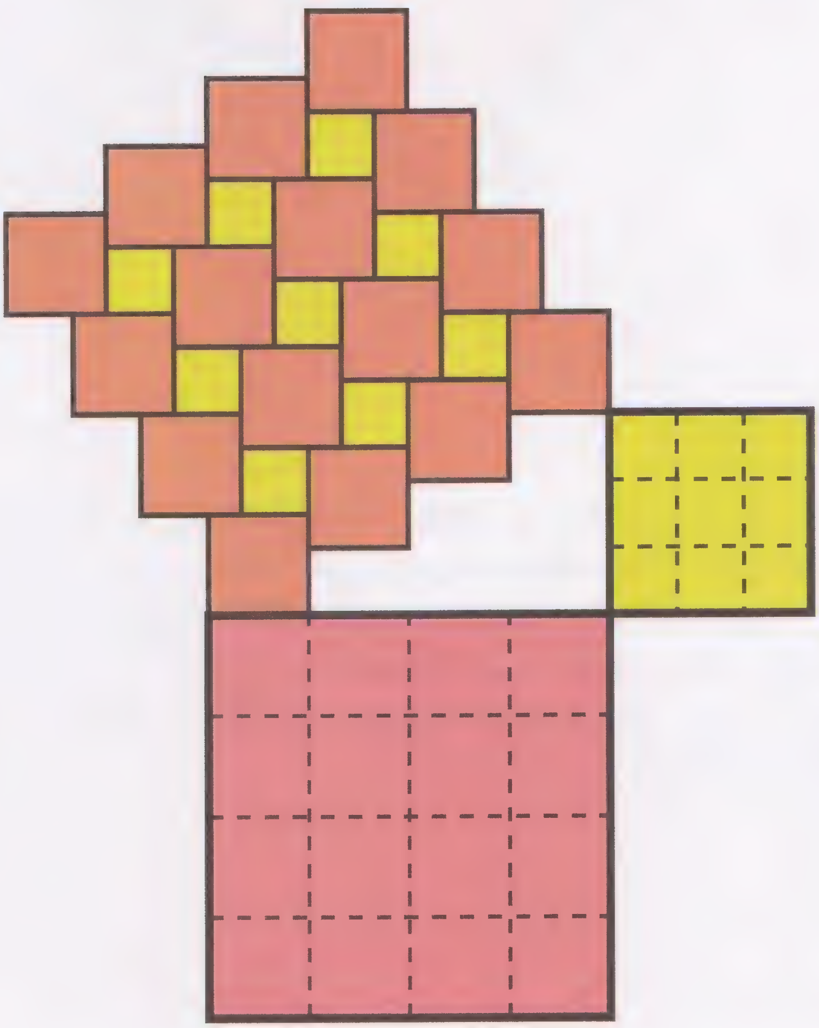
A first proof

Let ***A'*** and ***B'*** be two arbitrary squares. We look at Figure 5.5 for inspiration: dissect *A'* into 9 little congruent squares, and *B'* into 16 congruent squares, and join the 25 pieces together as in Figure 5.9. The toothed square obtained ***T'*** is equal in area (T) to the sum of the areas of the real squares *A'* and *B'*:

$$T = A + B.$$



a



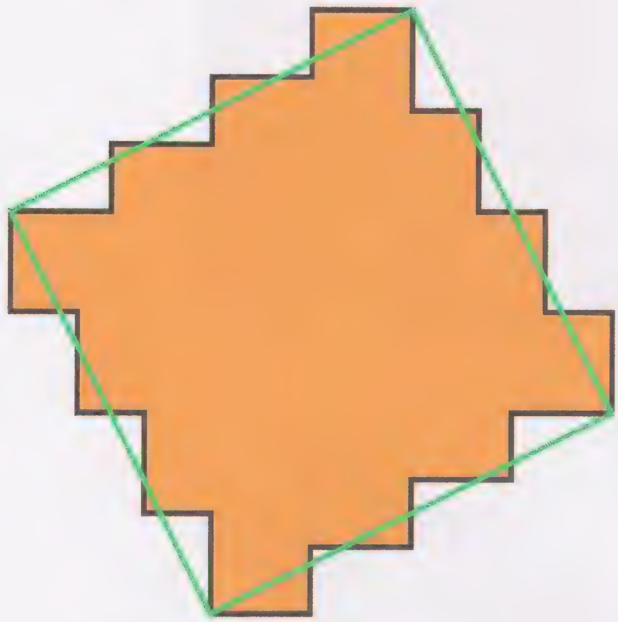
b

Figure 5.9

As, once again, the toothed square is easily transformed into a real square ***C'*** of the same area (see Figure 5.10), we arrive at

$$A + B = C,$$

that is, the Pythagorean Theorem in all its generality.



T = C
Figure 5.10

An infinity of proofs

Instead of dissecting *A'* and *B'* into 9 and 16 sub-squares, it is possible to dissect them into n^2 and $(n+1)^2$ congruent sub-squares for each integer value of *n* (positive). Figure 11 illustrates the case n = 14. To each value of ***n*** there corresponds a proof of the Pythagorean Proposition. [2] In other words, there exist an ***infinite number*** of demonstrations of this famous theorem.

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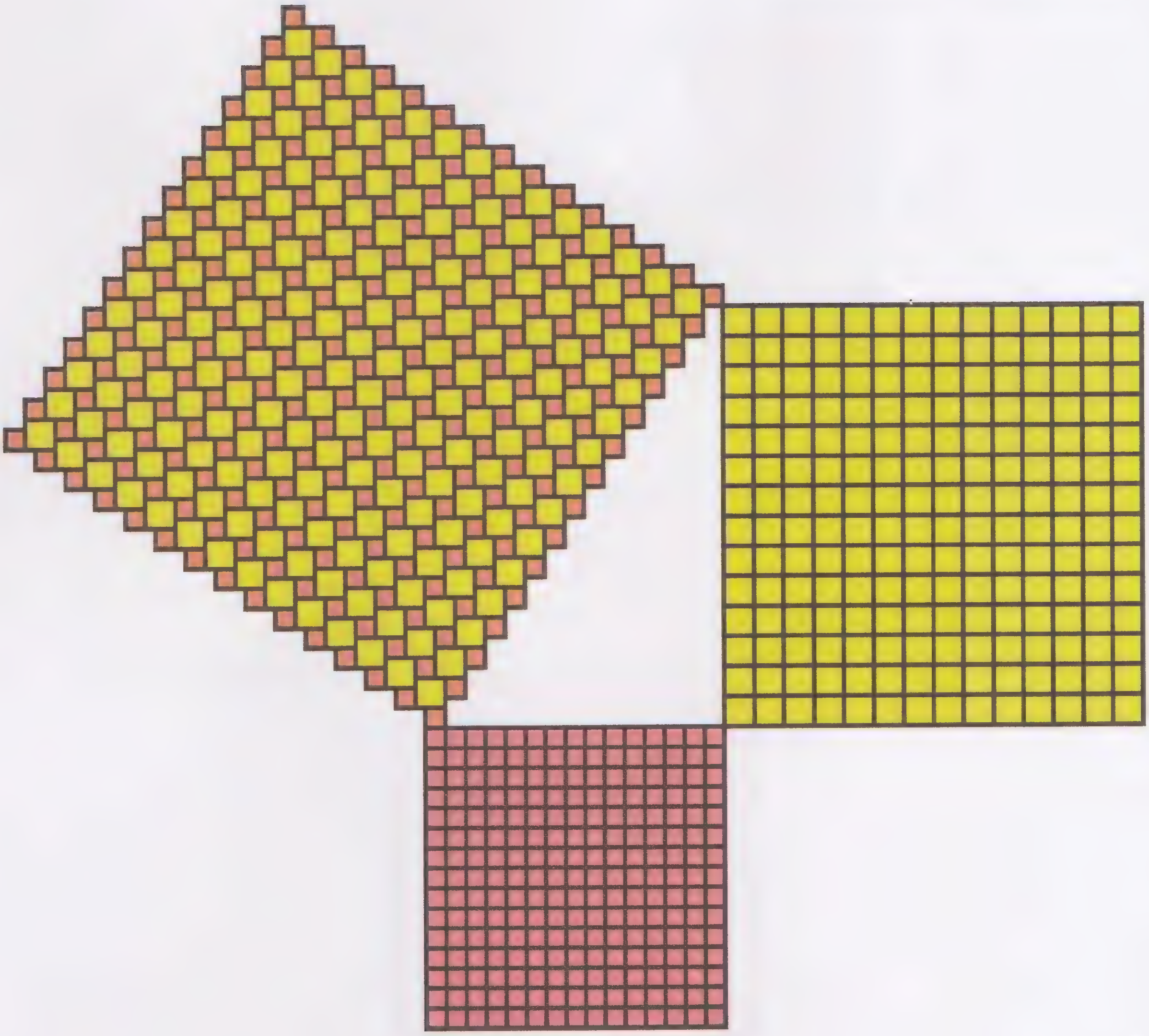
2 The author found these proofs in 1986. See (Gerdes, 1986d).

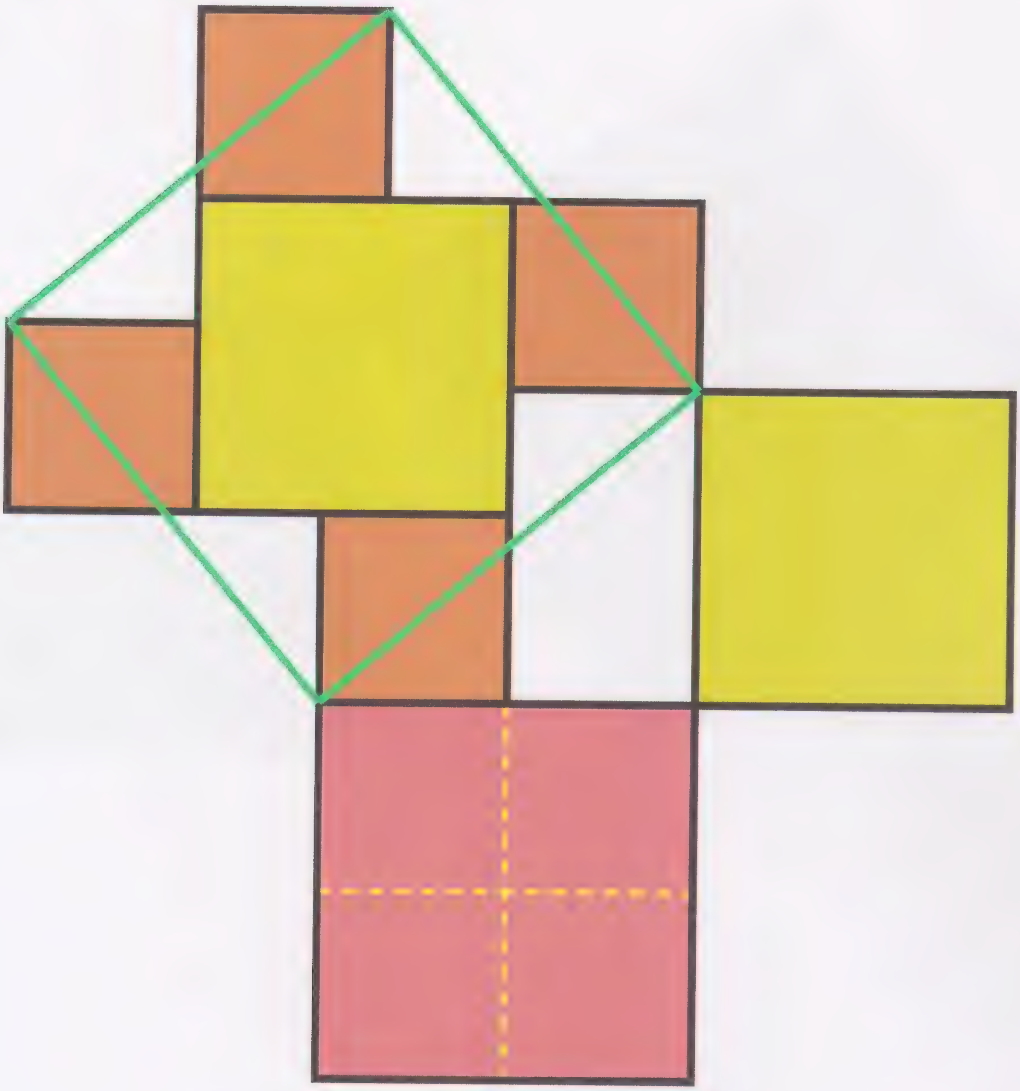


Case n = 14
Figure 5.11

For relatively high values of *n*, the truth of the Pythagorean Proposition is almost immediately visible. When we take the limit $n \to \infty$, we arrive at one more demonstration of the theorem. [3]

For n=1 we obtain a very short, easily understandable proof (see Figure 5.12).

3 Another proof by means of limits is presented in (Gerdes, 1986c).



Very easy proof in the case n = 1.
Figure 5.12

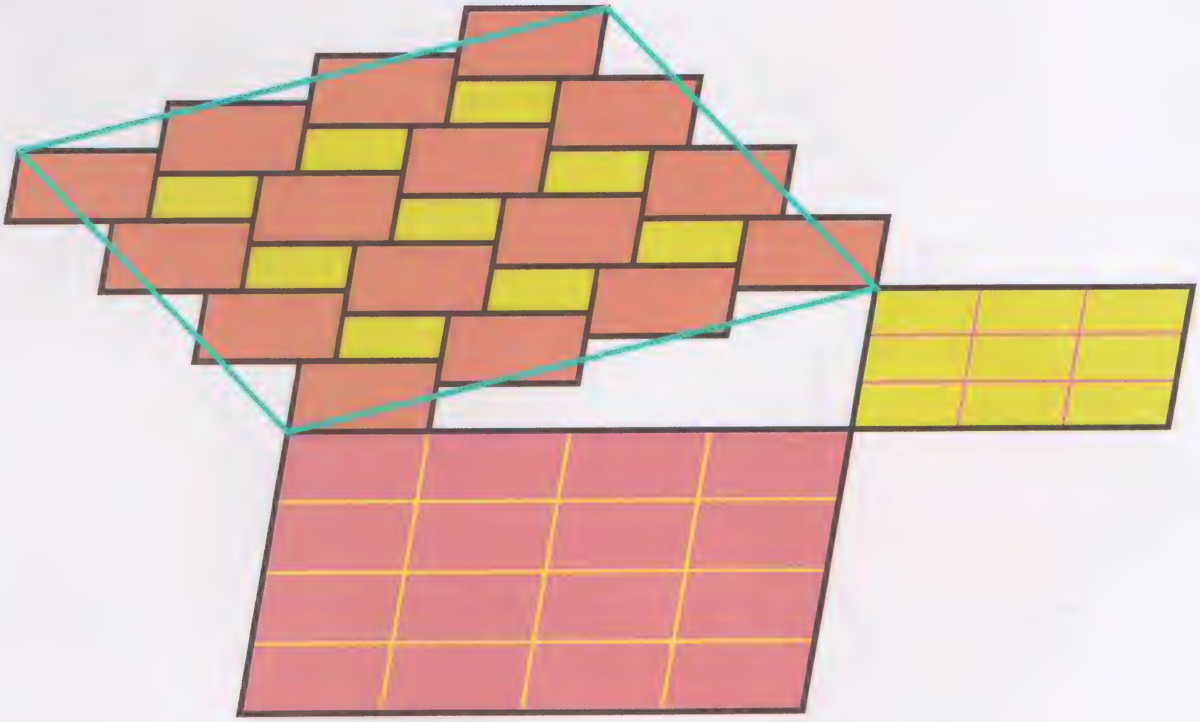
Pappus' Theorem

Analogously, Pappus' generalization of the Pythagorean Proposition for parallelograms can be proved in an infinite number of ways (Figure 5.13 illustrates the case n = 2).

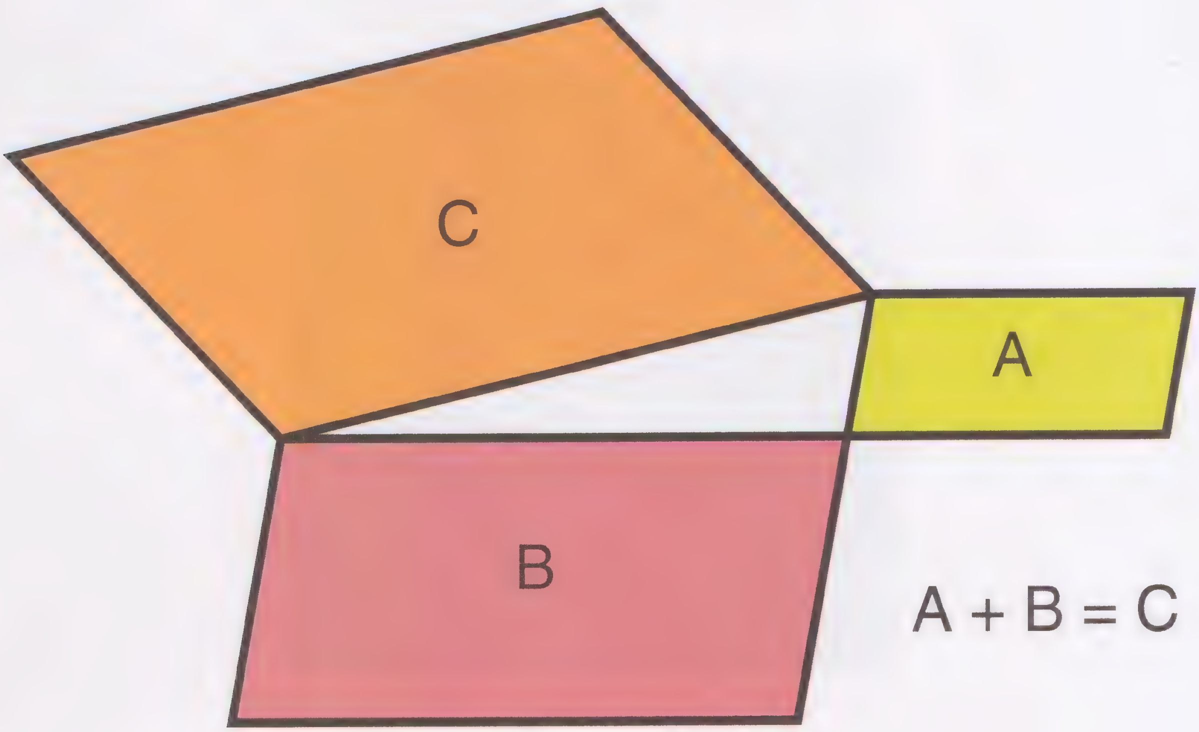
Example

Loomis' well known study ***The Pythagorean Proposition*** gives "...in all 370 different proofs, each calling for its specific figure" (1940; 1972, p. 269) and its author invites his audience to "Read and take your choice; or better, find a new, a different proof..." (p. 13). Our reflection on a widespread decorative motif led us not only to an alternative, active way of introducing the Pythagorean Proposition in

the classroom, but also of generating an ***infinite number*** of proofs of the same theorem.

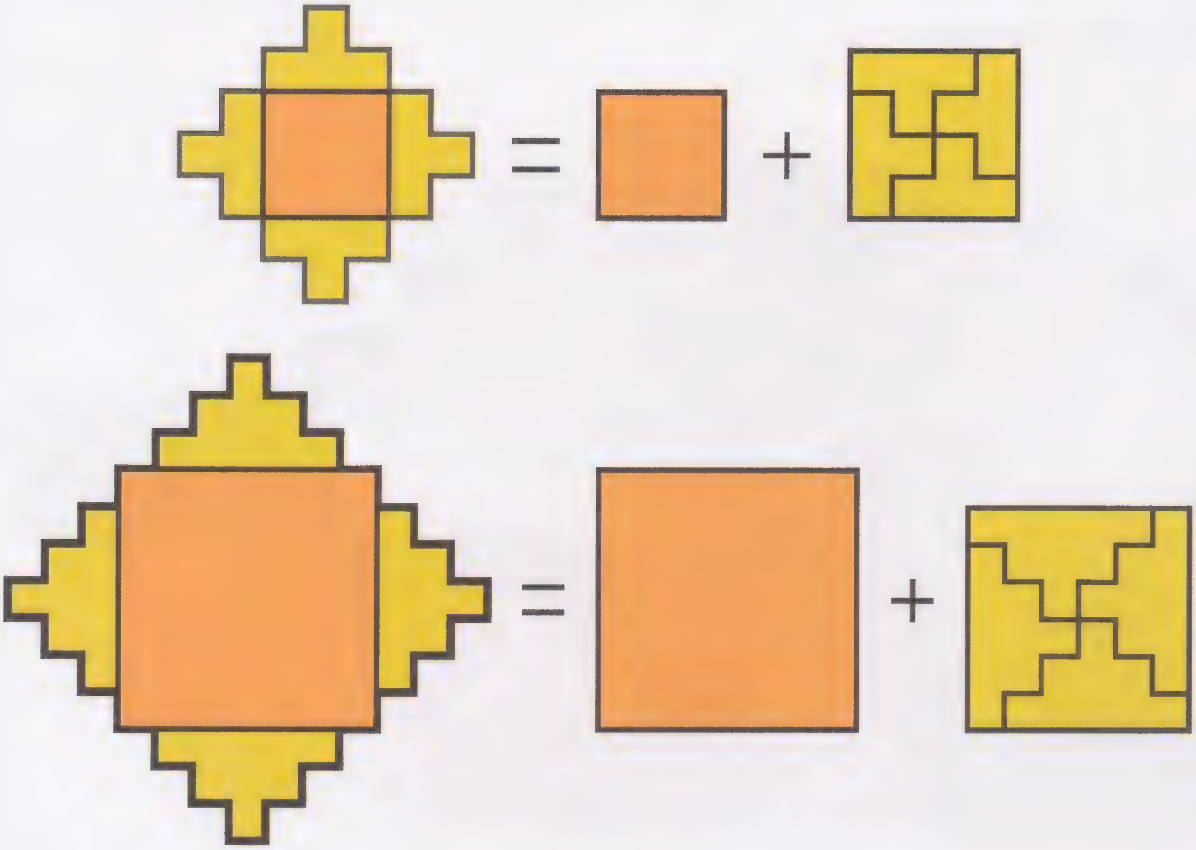


a



b

Figure 5.13



An alternative decomposition
Figure 5.14

Zulu basket (South Africa)
(Author's collection)

Chapter 6

FROM MAT WEAVING PATTERNS TO 'PYTHAGORAS' AND MAGIC SQUARES *

Figure 6.1

Problem:

Let us suppose that we weave with layers of dark coloured (vertical) strands and light coloured (horizontal) strands (Figure 6.1). How may we weave them over-and-under in such a way that we obtain an overall light coloured mat covered by a square grid of dark coloured dots (Figure 6.2)?

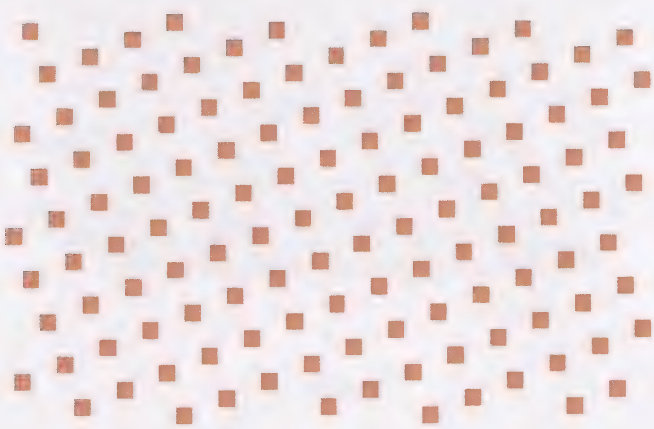


Figure 6.2

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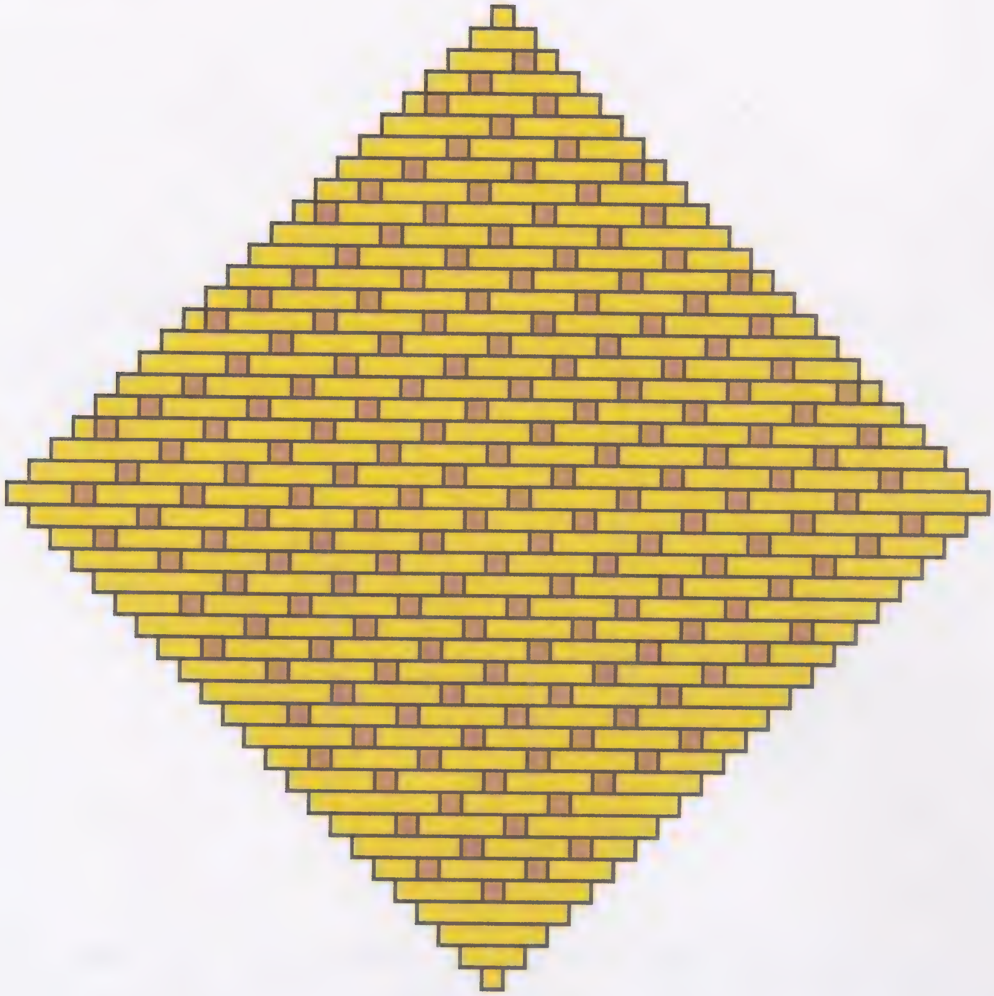
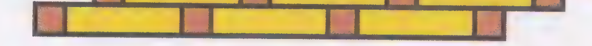
* Example presented during a lecture given at the 5th Symposium of the Southern African Mathematical Sciences Association, Maseru, Lesotho, December 15-19, 1986.



Weaving pattern on a Cokwe mat from Angola
a



b
Figure 6.3

The Cokwe solution

Figure 6.3 shows a solution found by Cokwe mat makers from north-eastern Angola: each dark coloured strand goes over one light coloured strand and then under four light coloured strands.

Diagram representation

Weaving patterns are usually represented by two-colour diagrams (often black-and-white) on squared paper, as Figure 6.4 illustrates in the case of the Cokwe fabric. Each yellow square corresponds to a place where the horizontal (yellow) strand passes over a vertical (brown) strand, and, inversely, we colour the square brown if the vertical (brown) strand passes over the horizontal (yellow) strand.

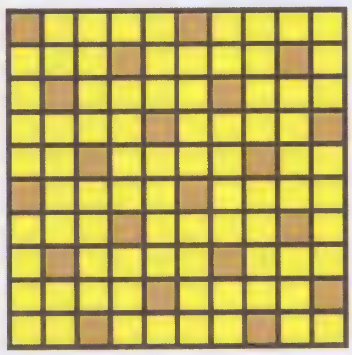


Figure 6.4

From a brown dot on the Cokwe diagram to the nearest dot on a neighbouring column, we move one unit to the right and two units downward, or, alternatively, one unit to the left and two upward (Figure 6.5a). From a brown dot to the nearest dot on a neighbouring row, we move two units to the right and one unit upward, or, alternatively, two units to the left and one downwards (Figure 6.5b). The direction of the successive dots in Figure 6.5a is perpendicular to the direction of successive dots in Figure 6.5b, as Figure 6.6 displays.

a b

Figure 6.5

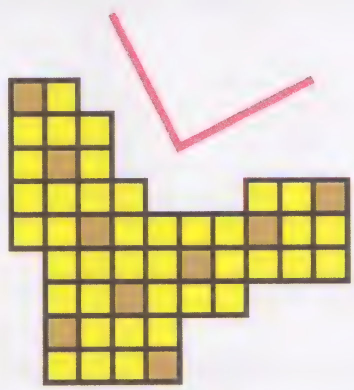


Figure 6.6

Other solutions

The diagrams in Figure 6.7 illustrate alternative solutions to the initial problem. If we also admit brown (square) dots composed of more than one unit square, other solutions become possible as Figure 6.8 shows.

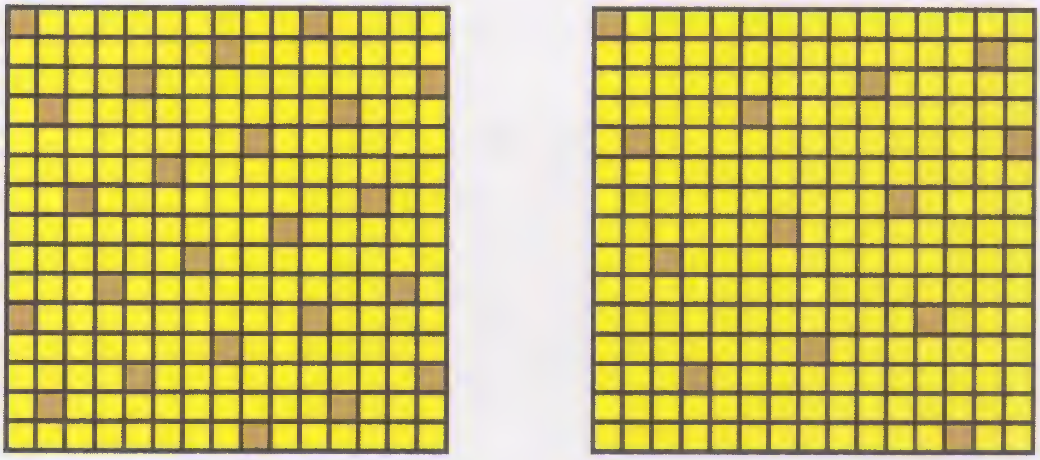


Figure 6.7

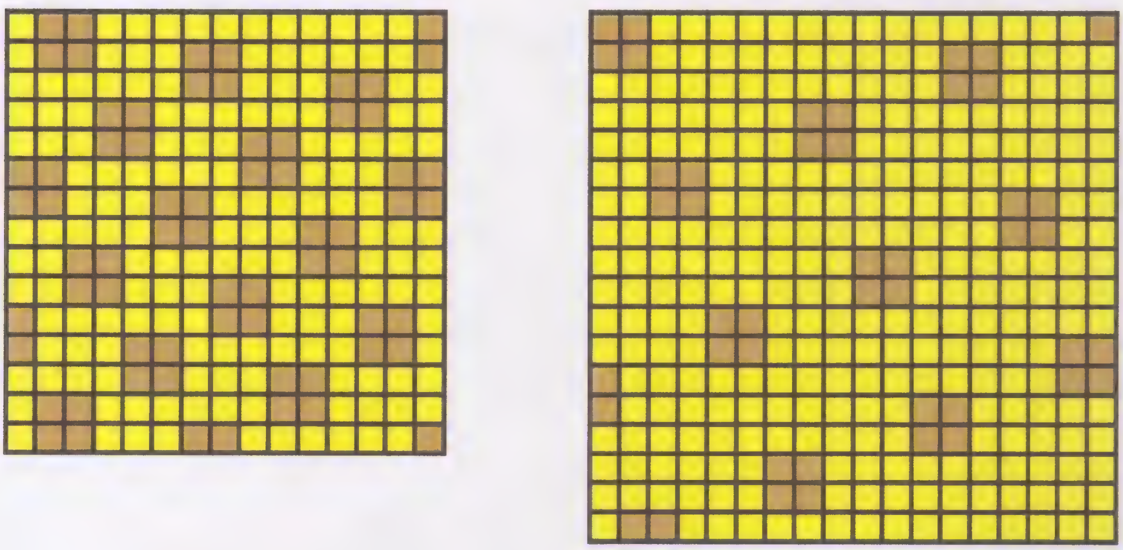


Figure 6.8

Discovering a theorem

All solutions (Figures 6.4, 6.7, 6.8) have the fact in common, that four neighbouring brown dots 'embrace' a yellow square (see Figure 6.9).

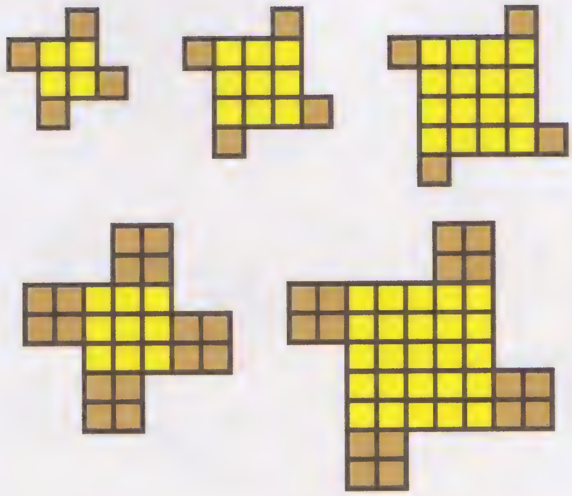


Figure 6.9

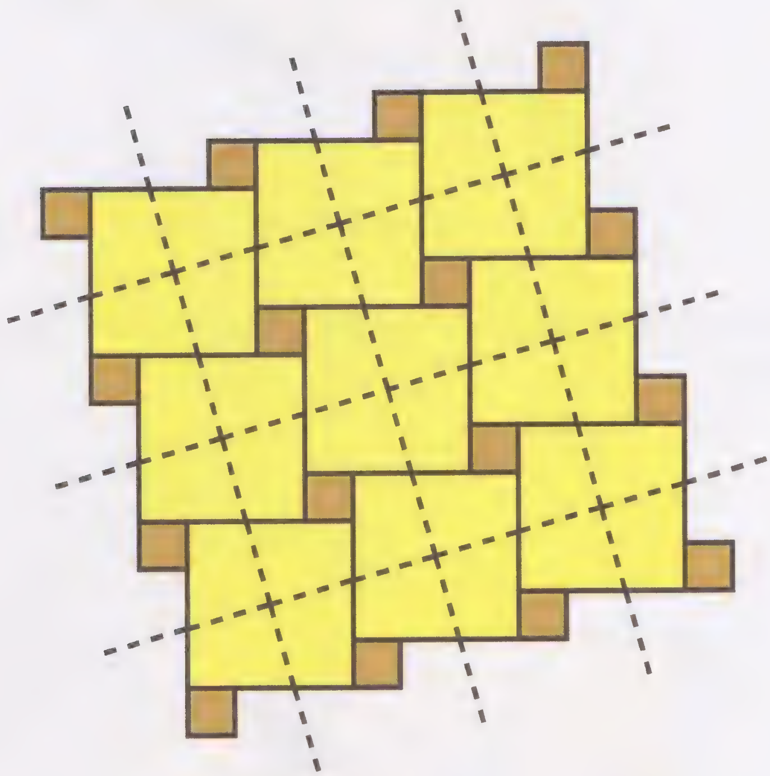


Figure 6.10

When we draw straight lines through the centres of these neighbouring yellow squares, a new square grid is obtained, this time an oblique one (see Figure 6.10). As each yellow square is divided by the lines of the new grid into four congruent quadrilaterals and, at the same time, each square of the oblique grid is composed of a brown square at its centre, surrounded by four such congruent quadrilaterals (see Figure 6.11), it follows that the area of an oblique square (**C**) is equal to the sum of the areas of one brown (**A**) and one yellow (**B**) square:

(1) $$A + B = C.$$

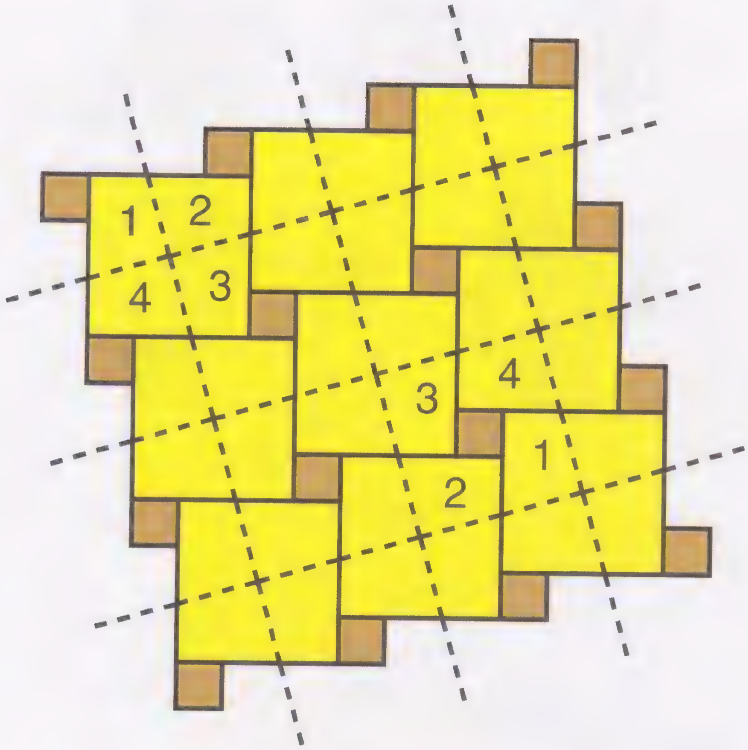


Figure 6.11

Which possible interpretations do there exist for the equality we have obtained?

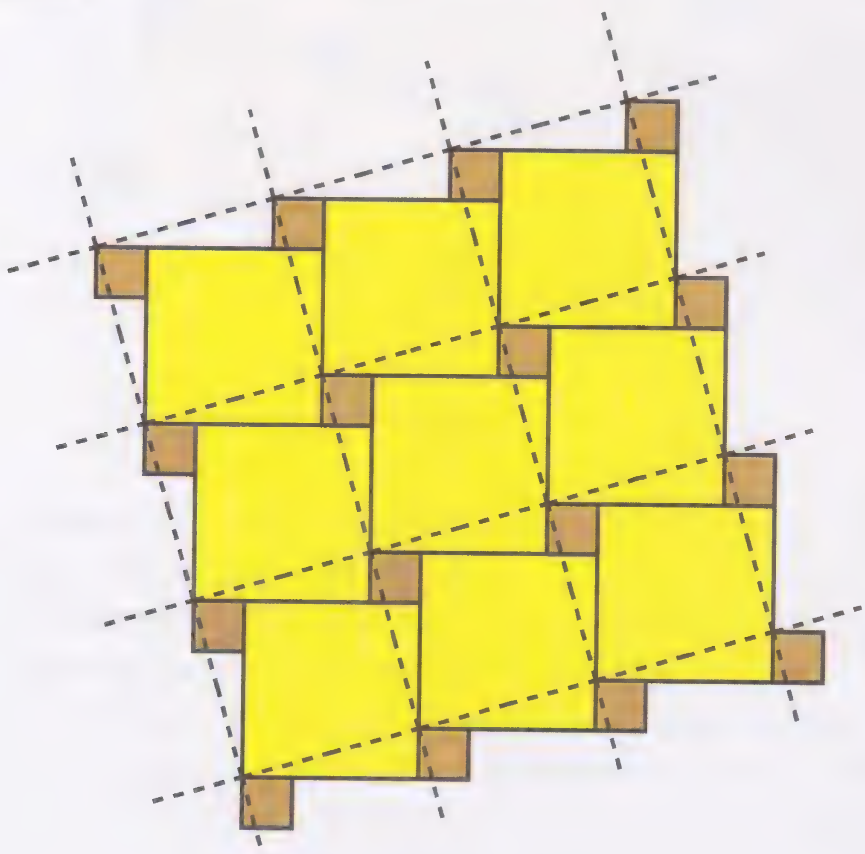


Figure 6.12

When we translate the oblique grid in such a way that its vertices coincide with vertices of the brown squares, as, for instance, in Figure 6.12, we arrive at:

$$a^2 + b^2 = c^2, \tag{2}$$

that is, at the Theorem of Pythagoras, where ***a***, ***b*** and ***c*** denote the lengths of the sides of the squares of area A, B and C (Figure 6.13).

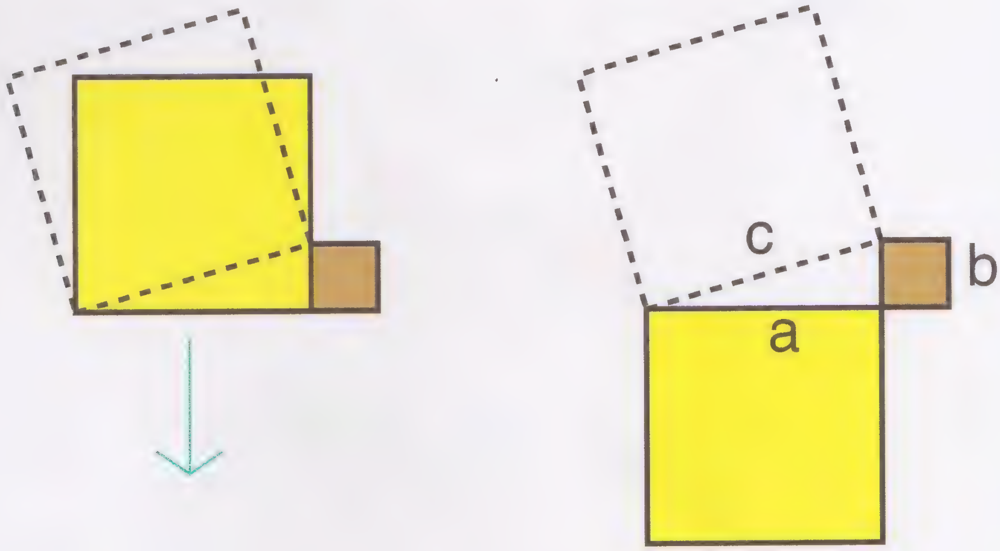


Figure 6.13

Questions:

Is it possible to fill up (3-dimensional) space with cubes of two sizes in a similar way?

Is it possible to generalize the Pythagorean Theorem to 3-dimensional space?

A variant

The pattern formed by four neighbouring brown dots and the 'embraced' yellow square (Figure 6.9) displays a rotational symmetry of 90°. This implies that four corresponding points are the vertices of a square. Figure 6.14 gives an interesting example, as the area of the new square obviously has the same area as the yellow square and the four brown squares together (see Figure 6.15). When we rearrange the yellow and brown squares as in Figure 6.16, we arrive once more at the Theorem of Pythagoras.

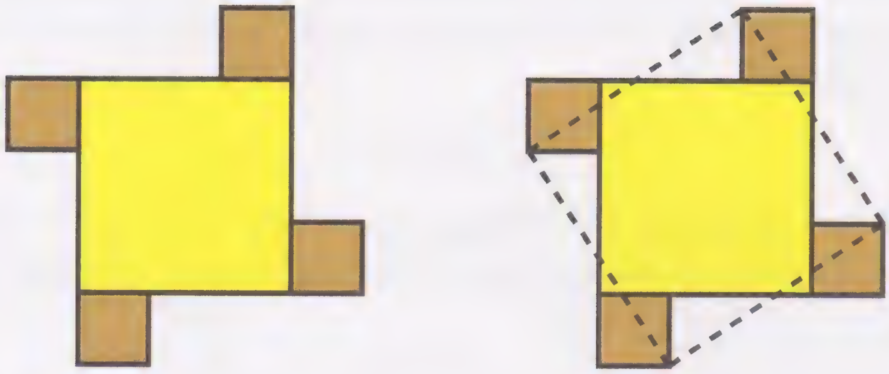


Figure 6.14

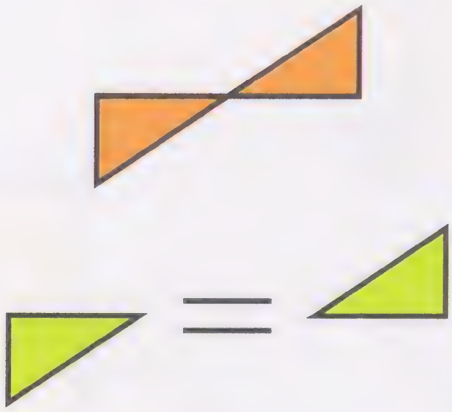


Figure 6.15

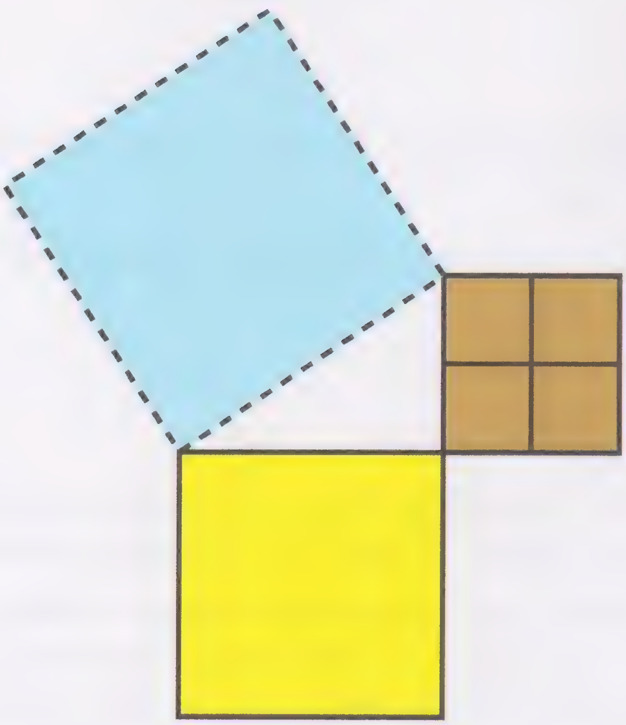


Figure 6.16

Question:

A similar reasoning may be developed in 3-dimensional space considering a cube 'surrounded' by small cubes?

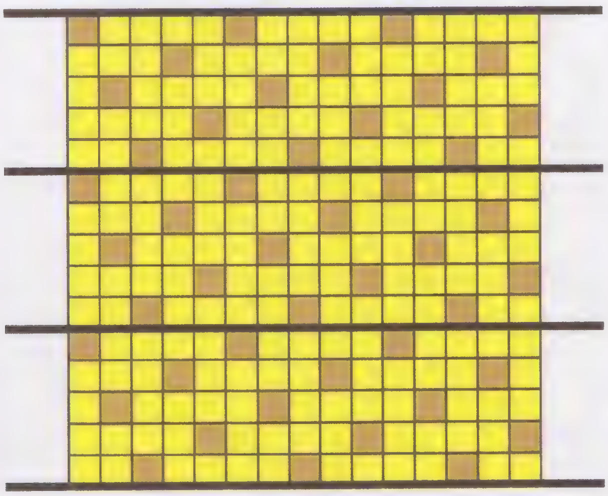


Figure 6.17

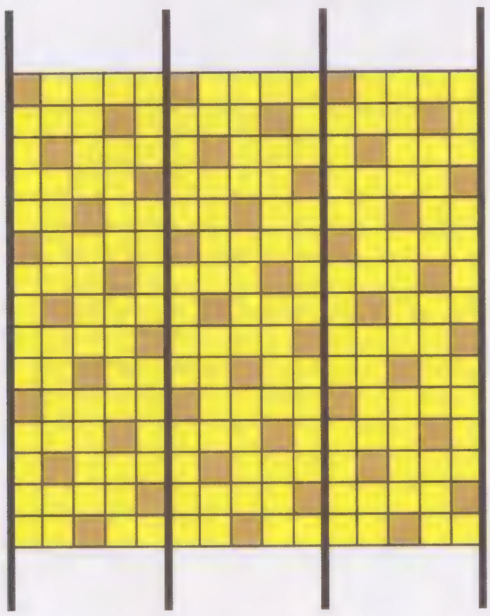


Figure 6.18

Periodicity and fundamental blocks

As the Cokwe artisans always weave the vertical brown strands over one horizontal yellow strand and then under four yellow strands, their two-colour design repeats itself in the vertical direction after 1+4 or 5 (horizontal) strands (see Figure 6.17). In the horizontal direction, each yellow strand passes under one brown strand and then over 4 brown strands. In the horizontal direction the pattern also repeats itself

after 1+4 or 5 strands (see Figure 6.18). The pattern has ***period*** 5 in both directions, that is, it may be considered as composed of equally coloured 5x5 blocks (see Figure 6.19). This fundamental 5x5 block contains just one little brown square in each row and each column.

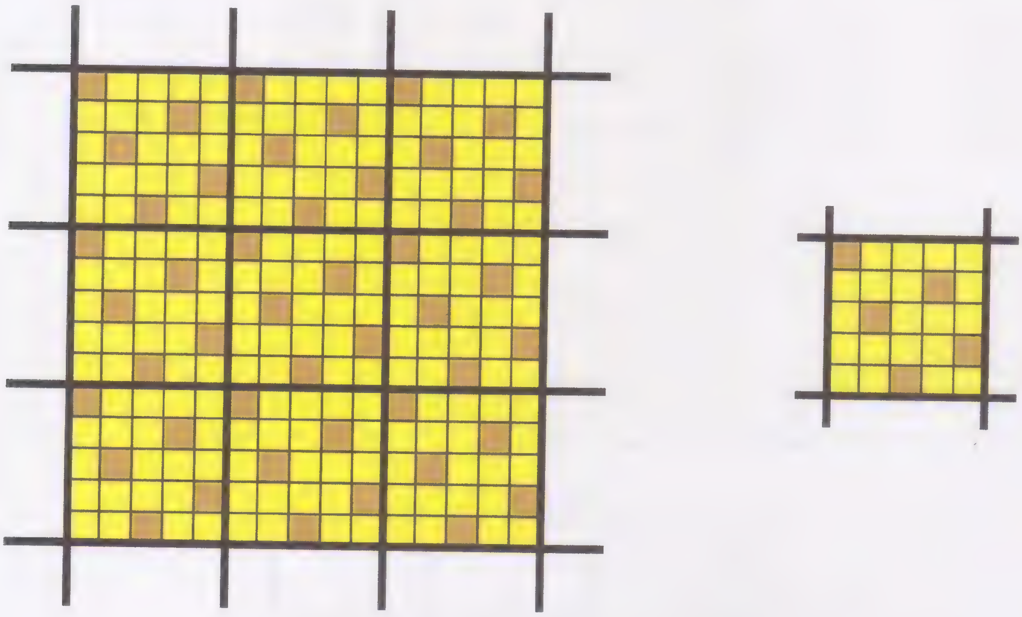


Figure 6.19

Problem

Let us enumerate, from the left to the right, 1, 2, 3, 4, 5 the successive brown squares of a fundamental block, as illustrated in Figure 6.20.

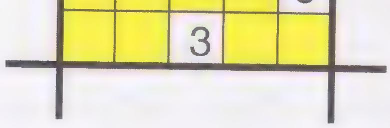


Figure 6.20

Is it possible to attribute, to each small square of a 5x5 block, a number 1, 2, 3, 4 or 5, in such a way that it is different from the numbers that appear in the same horizontal, vertical or diagonal line?

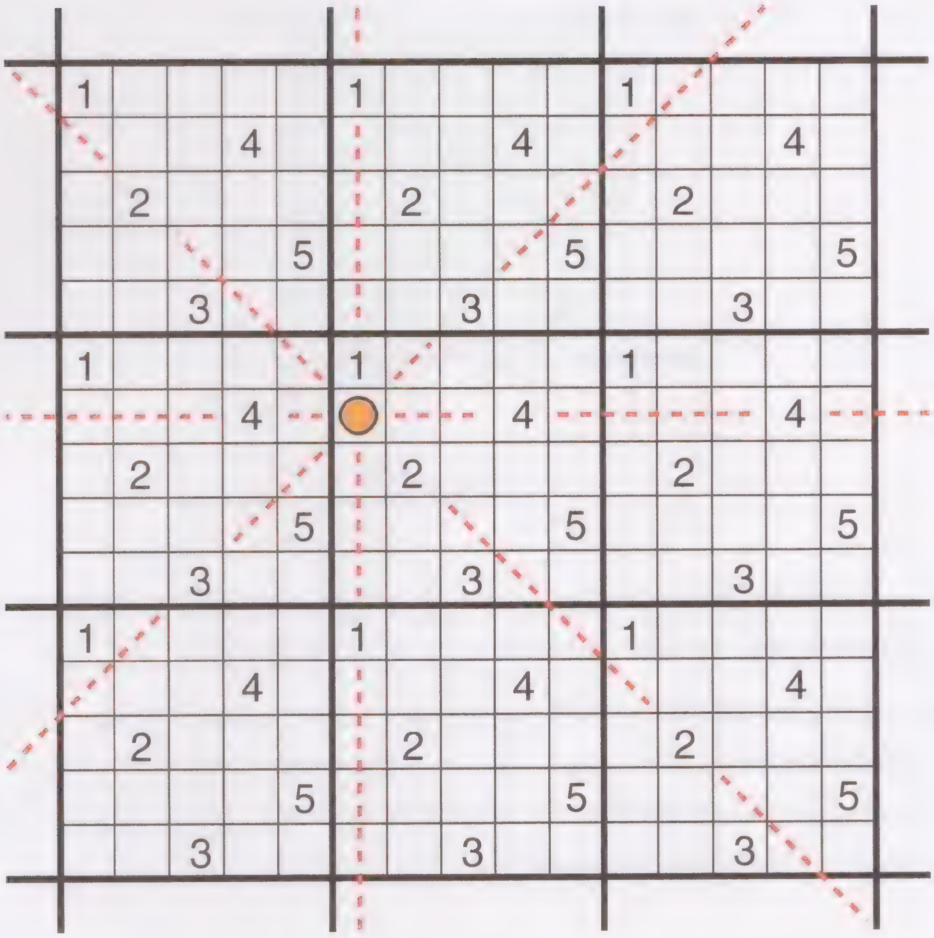


Figure 6.21

In the example of the small square marked by a circle in Figure 6.21, we see that there is only one possibility: we place only the number 5 in it.

A Latin square

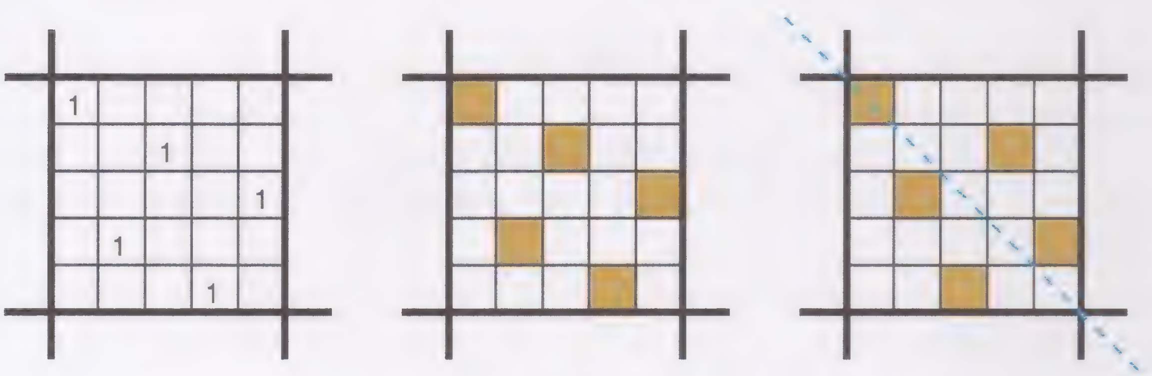
Doing the same for the other small squares, we obtain the 5x5 ***number square*** shown in Figure 6.22. In each row and each column the numbers 1, 2, 3, 4 and 5 appear precisely once. Such number squares are called ***Latin squares***.

1	4	2	5	3
5	3	1	4	2
4	2	5	3	1
3	1	4	2	5
2	5	3	1	4

Figure 6.22

We observe that the numbers 1, 2, 3, 4, and 5 appear in each column downwards in the sequence 5, 4, 3, 2, and 1.

The position of the 1's belonging to the Latin square constructed in the aforementioned way (see Figure 6.23a) is equal to the fundamental brown-and-yellow block (Figure 6.19b) after a reflection about its main diagonal as an axis (see Figures 6.23b and c).



a b c

Figure 6.23

The positions of the 2's, 3's or 4's are also interesting to note. The design formed by the 2's is the same as the pattern made up of the 1's after having been turned 90° to the left (see Figure 6.24). The 3's constitute the same pattern after the design of the 1's has been turned 90° to the right. The 4's form the pattern of the 1's after a rotation of 180°.

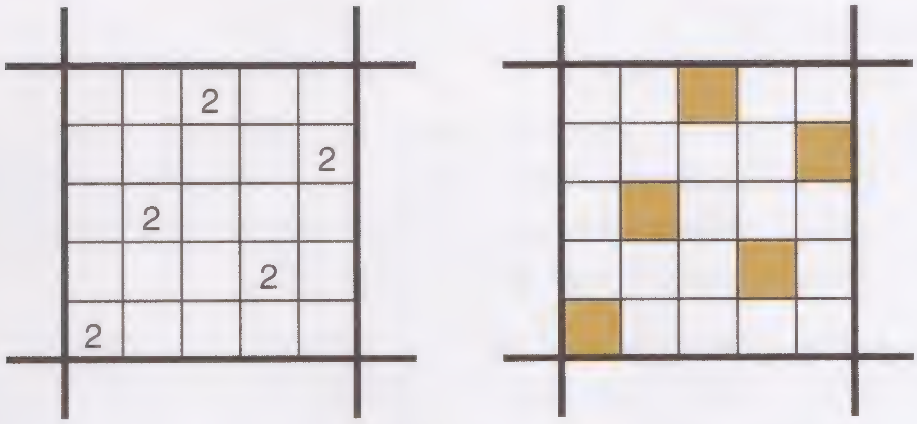


Figure 6.24

From a 1 to a 2 in the next column to the right, we always go two squares down. The successive 1, 2, 3, 4, and 5 once more constitute the initial design (see Figure 6.19b), turned 90° or 180° (Figures 6.25 c, e and b), with the case where the 5 is placed in the centre of the 5x5 number square as the only exception (see Figure 6.25d).

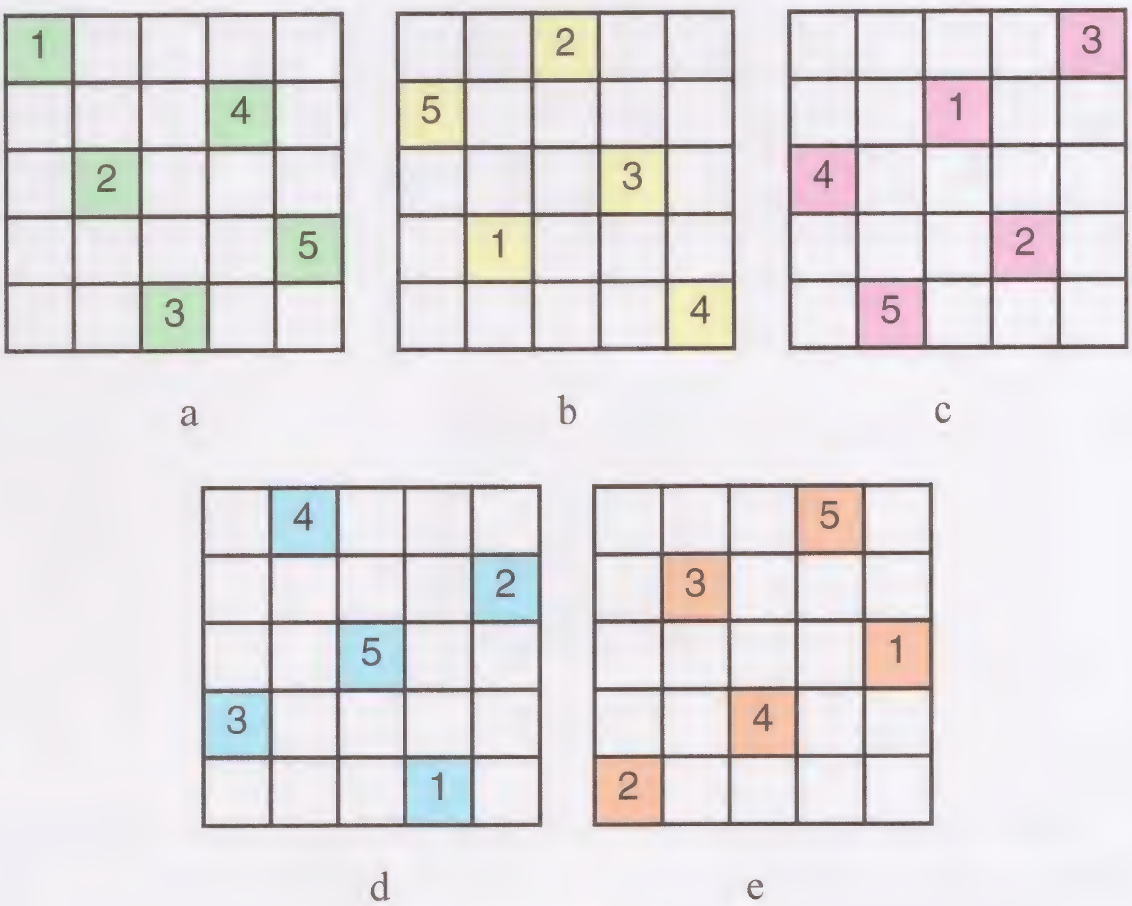


Figure 6.25

Magic squares

Our Latin square may be used to construct *magic squares*. Usually a magic square is defined as a set of integers in serial order, beginning with 1, arranged in square formation so that the sum total of each row, column and main diagonal is the same.

Our Latin square already has the property that the sum total of each row, column and main diagonal is the same. Therefore, if we add 5 to five numbers, one in each row and each column, adds 10 to five other numbers, one in each row and each column, and if we then add 15 and 20 in the same way, we obtain a magic square. Figure 6.26 gives an example. To the numbers of the block displayed in Figure 6.25b, we added 5, to the numbers of the block in Figure 6.25c we added 10, etc. The sum total of the rows, columns and main diagonal is 65.

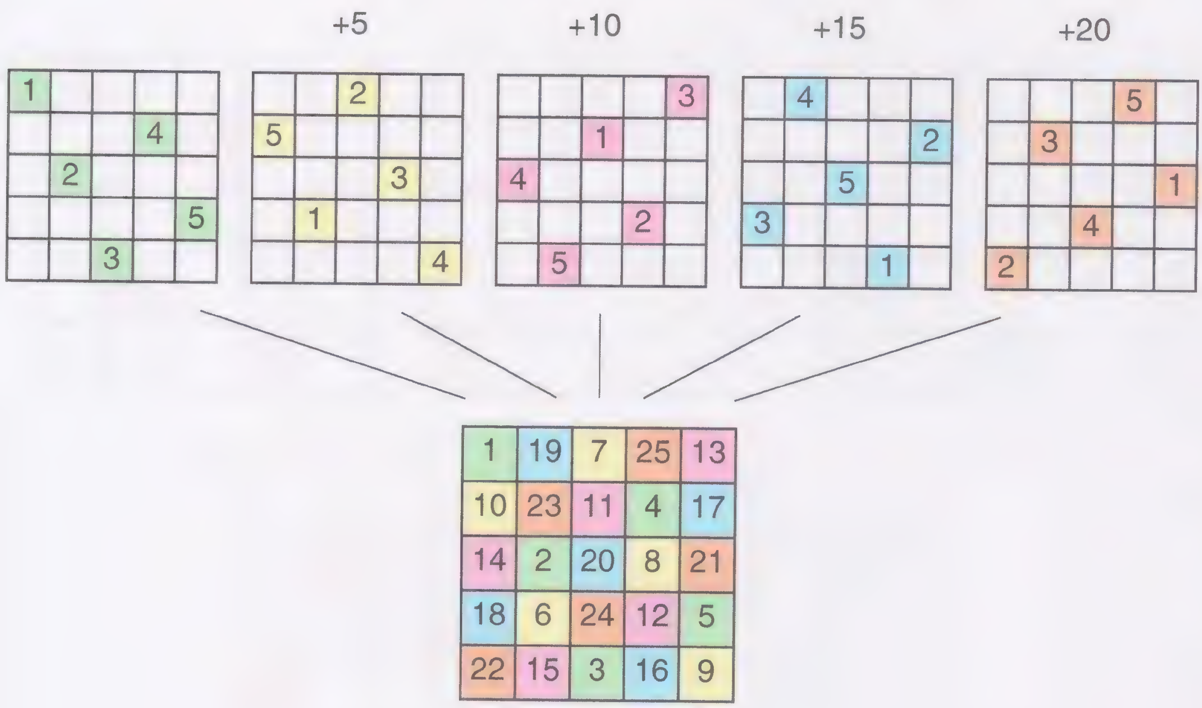


Figure 6.26

The magic square given in Figure 6.27 may be obtained by adding 5 to the numbers in block **a**, 10 to the numbers in block **e**, 15 to the numbers in block **c** and 20 to the numbers in block **b** of Figure 6.25.

6	4	22	15	18
25	13	16	9	2
19	7	5	23	11
3	21	14	17	10
12	20	8	1	24

Figure 6.27

These number squares have further properties. They are also ***diabolic*** or ***pandiagonal***, that is, the sum totals of the so-called 'broken diagonals' are equal to the same constant 65. 'Broken diagonals' are those that belong partly to a number square and partly to its equal neighbouring number square (see the examples in Figure 6.28).

1	19	7	25	13	1	19	7	25	13	1	19	7	25	13	1	19	7	25	13
10	23	11	4	17	10	23	11	4	17	10	23	11	4	17	10	23	11	4	17
14	2	20	8	21	14	2	20	8	21	14	2	20	8	21	14	2	20	8	21
18	6	24	12	5	18	6	24	12	5	18	6	24	12	5	18	6	24	12	5
22	15	3	16	9	22	15	3	16	9	22	15	3	16	9	22	15	3	16	9

65 65

7+23+14+5+16=65 7+4+21+18+15=65 1+17+8+24+15=65, etc.

Figure 6.28

When we join several equal magic squares of this type, as in Figure 6.29a, and then displace the 5x5 square as in the example in Figure 6.29b, we obtain new magic squares (see Figure 6.29c).

6	4	22	15	18	6	4	22	15	18
25	13	16	9	2	25	13	16	9	2
19	7	5	23	11	19	7	5	23	11
3	21	14	17	10	3	21	14	17	10
12	20	8	1	24	12	20	8	1	24
6	4	22	15	18	6	4	22	15	18
25	13	16	9	2	25	13	16	9	2
19	7	5	23	11	19	7	5	23	11
3	21	14	17	10	3	21	14	17	10
12	20	8	1	24	12	20	8	1	24

a

6	4	22	15	18	6	4	22	15	18
25	13	16	9	2	25	13	16	9	2
19	7	5	23	11	19	7	5	23	11
3	21	14	17	10	3	21	14	17	10
12	20	8	1	24	12	20	8	1	24
6	4	22	15	18	6	4	22	15	18
25	13	16	9	2	25	13	16	9	2
19	7	5	23	11	19	7	5	23	11
3	21	14	17	10	3	21	14	17	10
12	20	8	1	24	12	20	8	1	24

b

8	1	24	12	20
22	15	18	6	4
16	9	2	25	13
5	23	11	19	7
14	17	10	3	21

c

Figure 6.29

The magic square in Figure 6.27 has the additional property that each number on one of the four axes of the square, added to the number symmetrically opposite the square's centre, yields the same: 12+18 = 14+16 = 7+23 = ... = 30 (see Figure 6.30).

6				18
	13	16	9	
19	7		23	11
	21	14	17	
12				24

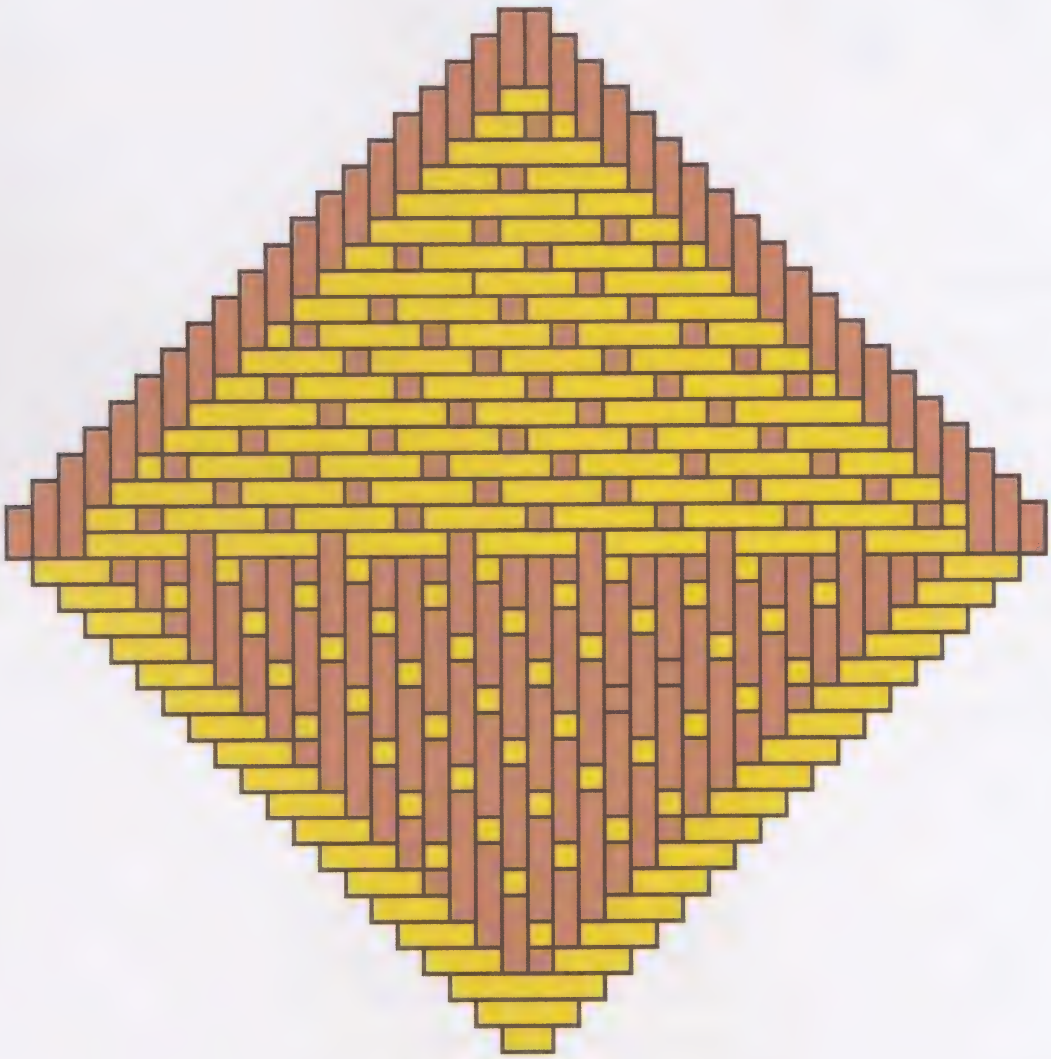
Figure 6.30

13	25	7	19	1
17	4	11	23	10
21	8	20	2	14
5	12	24	6	18
9	16	3	15	22

Figure 6.31

When we reflect the magic square in Figure 6.26 about its

vertical axis, we obtain one of the magic squares (Figure 6.31) presented by ***Muhammad ibn Muhammad al Fullani***, an early eighteenth century astronomer and mathematician from Katsina (now northern Nigeria), in his work ***A treatise on the magical use of the letters of the alphabet***, written in Arabic (cf. Zaslavsky, 1973, p. 138-151).

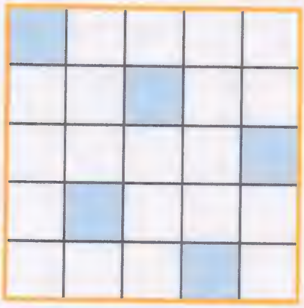


Kuba weaving design
Figure 6.32

An alternative Kuba solution

An alternative to the Cokwe solution to the initial mat weaving problem is presented in the Kuba (Central Congo) pattern illustrated in Figure 6.32b. Figure 6.32a displays a detail. It has the 5x5 block shown in Figure 6.33a as the fundamental block. In its turn, this fundamental block leads to the Latin square displayed in Figure 6.33c after enumerating the brown dots (Figure 6.33b). This Latin square

may be used to construct many magic squares.



a

1				
		3		
				5
	2			
			4	

b

1	4	2	5	3
2	5	3	1	4
3	1	4	2	5
4	2	5	3	1
5	3	1	4	2

c

Figure 6.33

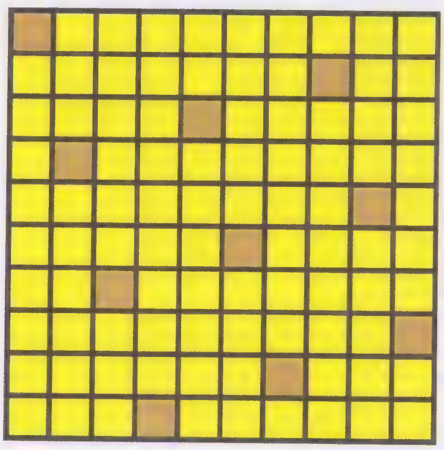
Questions:

How many magic squares may be constructed from this Latin square?

How many 5x5 Latin squares may be constructed?

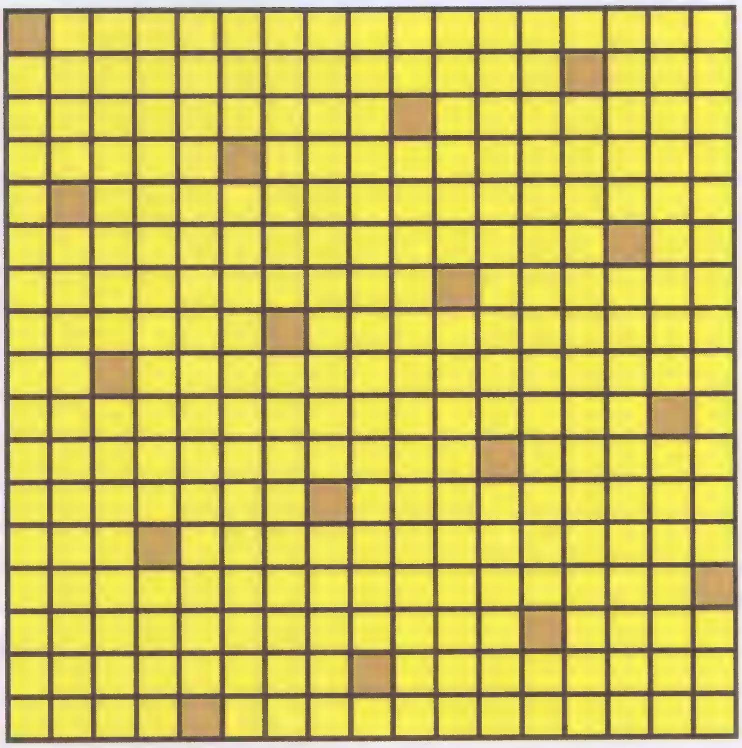
Solutions with other periods

The alternative solutions in Figure 6.7 have period 10 and 17 respectively. Figure 34 displays their fundamental blocks.



a

Figure 6.34



b

Figure 6.34

As before, by enumerating the successive small squares in the case of the 17x17 block (Figure 6.35), we may construct Latin and magic squares. Figures 6.36 and 6.37 give an example.

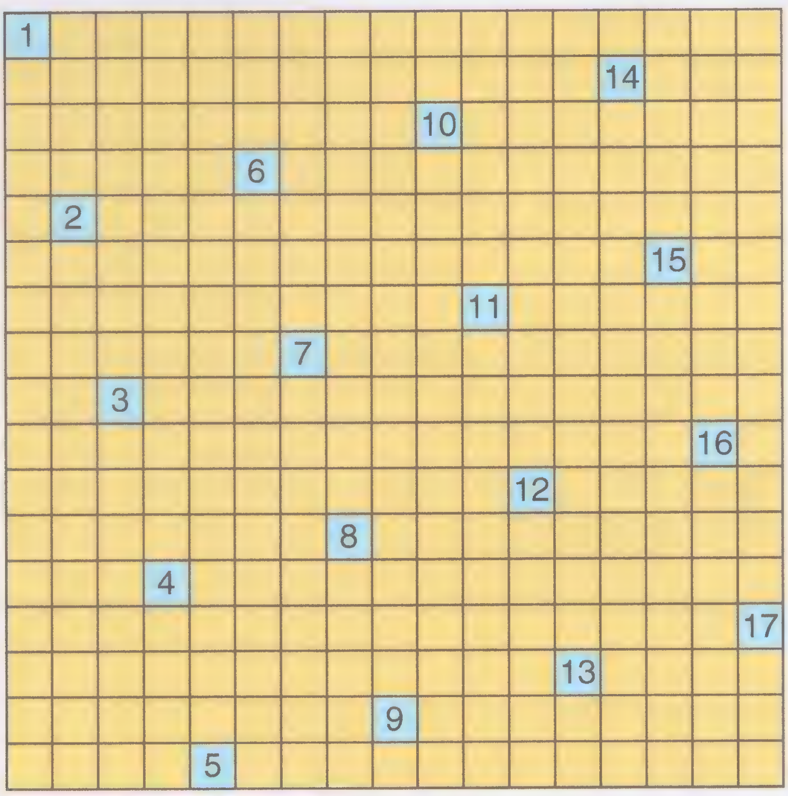


Figure 6.35

1	4	7	10	13	16	2	5	8	11	14	17	3	6	9	12	15
9	12	15	1	4	7	10	13	16	2	5	8	11	14	17	3	6
17	3	6	9	12	15	1	4	7	10	13	16	2	5	8	11	14
8	11	14	17	3	6	9	12	15	1	4	7	10	13	16	2	5
16	2	5	8	11	14	17	3	6	9	12	15	1	4	7	10	13
7	10	13	16	2	5	8	11	14	17	3	6	9	12	15	1	4
15	1	4	7	10	13	16	2	5	8	11	14	17	3	6	9	12
6	9	12	15	1	4	7	10	13	16	2	5	8	11	14	17	3
14	17	3	6	9	12	15	1	4	7	10	13	16	2	5	8	11
5	8	11	14	17	3	6	9	12	15	1	4	7	10	13	16	2
13	16	2	5	8	11	14	17	3	6	9	12	15	1	4	7	10
4	7	10	13	16	2	5	8	11	14	17	3	6	9	12	15	1
12	15	1	4	7	10	13	16	2	5	8	11	14	17	3	6	9
3	6	9	12	15	1	4	7	10	13	16	2	5	8	11	14	17
11	14	17	3	6	9	12	15	1	4	7	10	13	16	2	5	8
2	5	8	11	14	17	3	6	9	12	15	1	4	7	10	13	16
10	13	16	2	5	8	11	14	17	3	6	9	12	15	1	4	7

Latin square of dimensions 17 x 17
Figure 6.36

Questions:

Try to construct 17x17 Latin and magic squares.

How many 17x17 Latin squares may be constructed?

On the basis of a given 17x17 Latin square, how many magic squares may be constructed?

Try to construct 10x10 magic squares. Can the same method be used?

Try to construct 37x37 magic squares.

For which integers *n*, can we construct **n x n** magic squares in the described way?

1	38	75	112	149	186	206	243	280	28	65	102	122	159	196	233	270
145	182	219	239	276	24	61	98	135	155	192	229	266	14	51	71	108
289	20	57	94	131	168	188	225	262	10	47	84	104	141	178	215	252
127	164	201	238	258	6	43	80	117	137	174	211	248	285	33	53	90
271	2	39	76	113	150	187	207	244	281	29	66	86	123	160	197	234
109	146	183	220	240	277	25	62	99	136	156	193	230	267	15	35	72
253	273	21	58	95	132	169	189	226	263	11	48	85	105	142	179	216
91	128	165	202	222	259	7	44	81	118	138	175	212	249	286	34	54
235	272	3	40	77	114	151	171	208	245	282	30	67	87	124	161	198
73	110	147	184	221	241	278	26	63	100	120	157	194	231	268	16	36
217	254	274	22	59	96	133	170	190	227	264	12	49	69	106	143	180
55	92	129	166	203	223	260	8	45	82	119	139	176	213	250	287	18
199	236	256	4	41	78	115	152	172	209	246	283	31	68	88	125	162
37	74	111	148	185	205	242	279	27	64	101	121	158	195	232	269	17
181	218	255	275	23	60	97	134	154	191	228	265	13	50	70	107	144
19	56	93	130	167	204	224	261	9	46	83	103	140	177	214	251	288
163	200	237	257	5	42	79	116	153	173	210	247	284	32	52	89	126

Magic square of dimensions 17 x 17
Figure 6.37

Arithmetic modulo 5

Latin squares constitute an interesting context for the introduction of ***arithmetic modulo 5***.

Enumerating modulo 5 means counting 1, 2, 3, 4, 5, 1, 2, 3, 4, 5, 1, 2, 3, 4, 5, etc. (or: 1, 2, 3, 4, 0, 1, 2, 3, 4, 0, etc.). In order to add, for instance, 3 plus 4 modulo 5, we have to go from 3 four places to the right in the modulo 5 number series:

3, 4, 5, 1, 2.

We arrive at 2, that is, $3+4 \equiv 2$ (modulo 5).

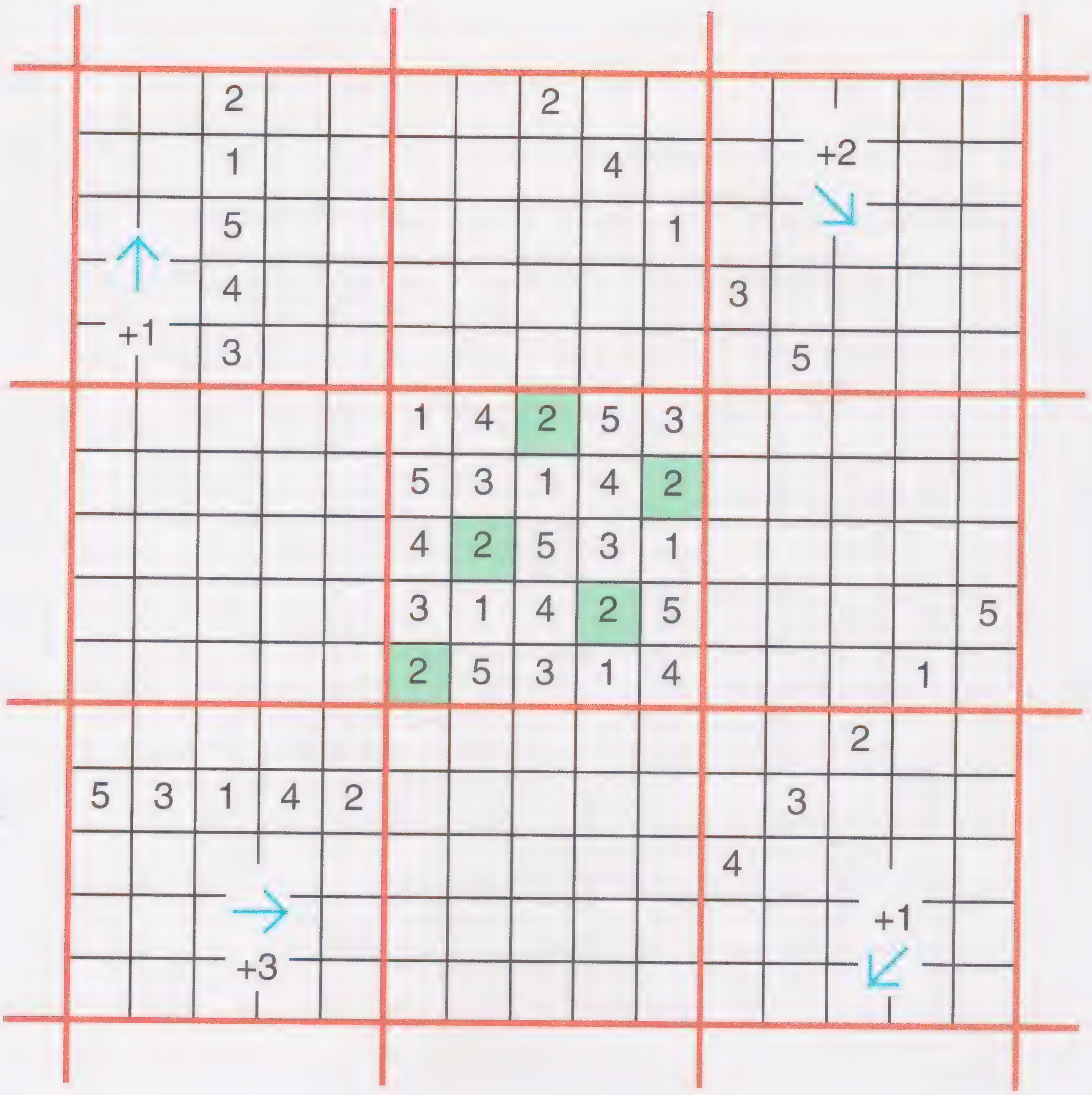


Figure 6.38

Now consider the Latin square in Figure 6.22. In order to go one place to the right in this Latin square, we always have to add 3 modulo 5. In order to go one place up, we have to add 1 modulo 5 (see Figure 6.38). When we go one place diagonally down to the right, what will be the end effect? First we may go one place to the right (+3) and then one place vertically downward (-1); the result is +2. When we move one place diagonally downward to the left, we go, for instance, first one place vertically downward (-1) and then one place left (-3); the end result is -4, but -4 ≡ +1 modulo 5 (see Figure 6.39).

Going from 2 to 5 in Figure 6.39, we add 3. If we pass through 4, we add first 1 and then two times 3, and from 4 to 5, we add first 3 and then subtract two times 1. Comparing the two ways, we see:

$$(1+2.3) + (3+ 2x[-1]) \equiv 3 \quad \text{(modulo 5)}$$

that is,

$$7+1 \equiv 3 \text{ (modulo 5)}.$$

In the same way, Latin squares of order 10, 17, etc. may be used to introduce the arithmetic modulo 10, modulo 17, etc.

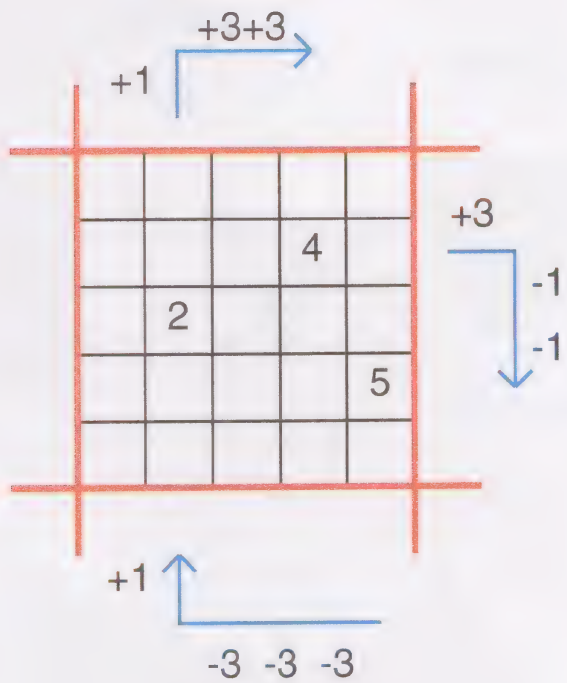


Figure 6.39

Questions:

Taking modulo *n* the integers in the **n x n** magic squares we constructed before, we return to the Latin square from which we started. What happens in the case of an arbitrary magic square? Do we always obtain a Latin square, when we take its numbers modulo its order? How can you prove your answer?

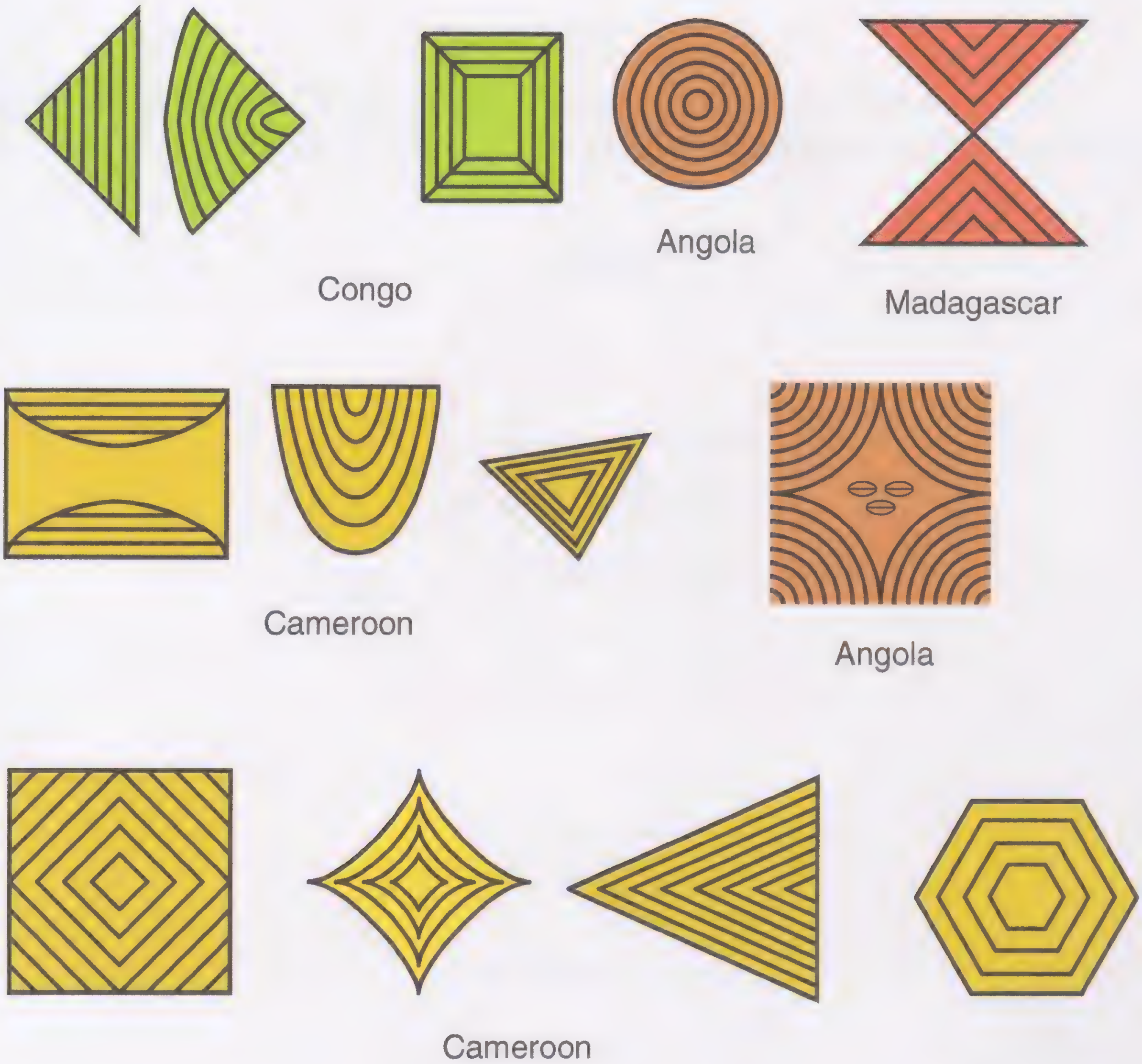


Figure 7.1

Chapter 7

A NEW PROOF BY MEANS OF LIMITS *

'Repetitive' patterns in African art

'Repetitive' patterns of the type shown in Figure 7.1 are very common in African art.

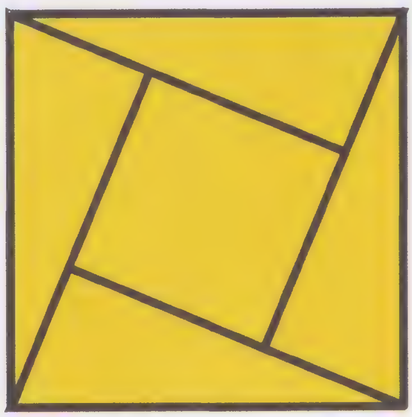


Figure 7.2

Constructing a new repetitive pattern

We may take other designs, like the Kuba elephants' defence design that we saw already in chapter 3 (see Figure 7.2). Both its borders (external and internal) are squares. The internal border square may be taken as the external border square of a second elephants' defence design, but this time of more reduced dimensions than the first one. The internal border square of the second elephants' defence design may in its turn be taken as the external border square of a third elephants' defence design, further reduced, etc. Figure 7.3 illustrates the first three elephants' defence designs constructed in this way. The vertices of the successive external border squares lie on four spirals

* The author found this proof in 1986. See (Gerdes, 1986c).

that pass through the centre of the original square (see Figure 7.4).

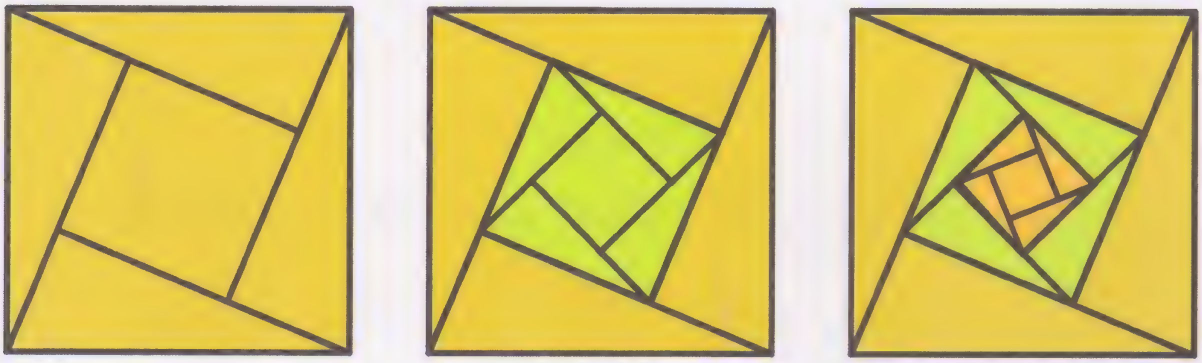


Figure 7.3

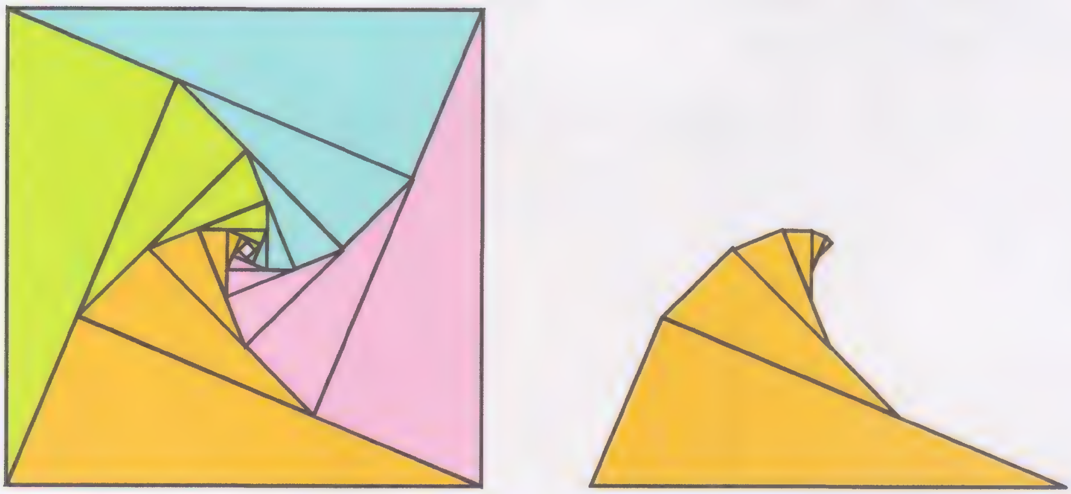
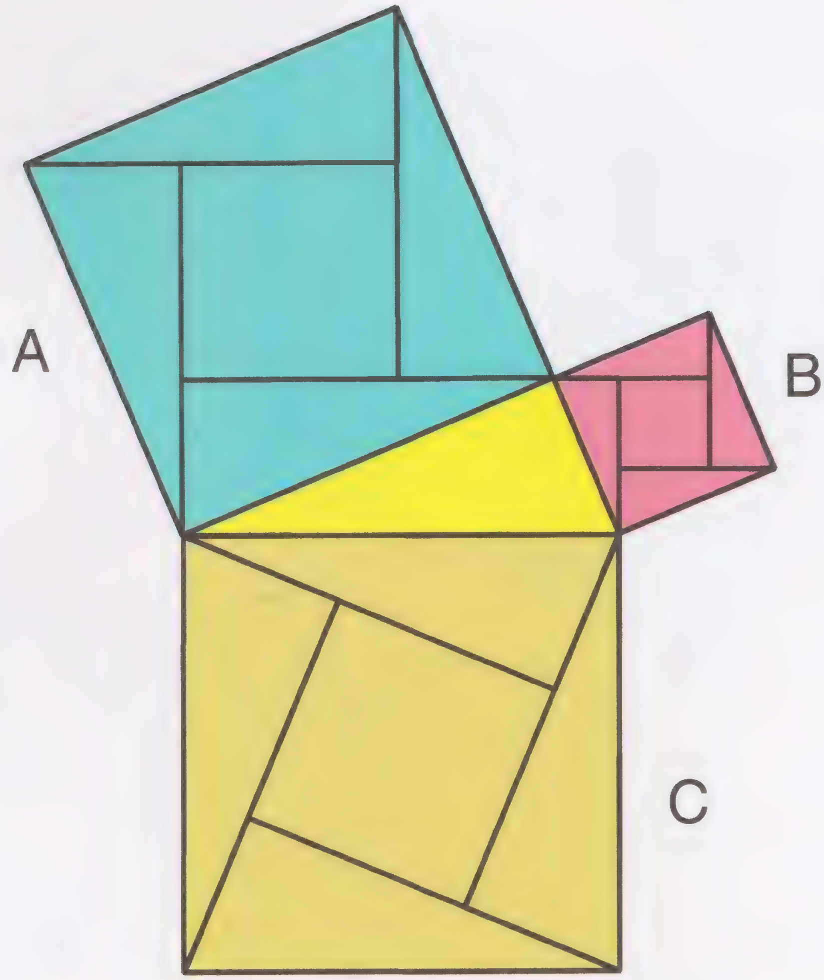


Figure 7.4

Right-angled triangles and the repetitive elephants' defence design

What happens if we consider elephants' defence designs constructed on the sides of a right-angled triangle (see Figure 7.5)?

Let **A**, **B** and **C** be the areas of the squares on the sides; $\mathbf{A_1}$, $\mathbf{B_1}$ and $\mathbf{C_1}$ the areas of the little squares that appear in their interior and $\mathbf{P_1}$, $\mathbf{Q_1}$ and $\mathbf{R_1}$ the areas of the surrounding triangles.



a

Figure 7.5

What may be said about A, B, and C? Which will be larger: C, or the sum of A and B?

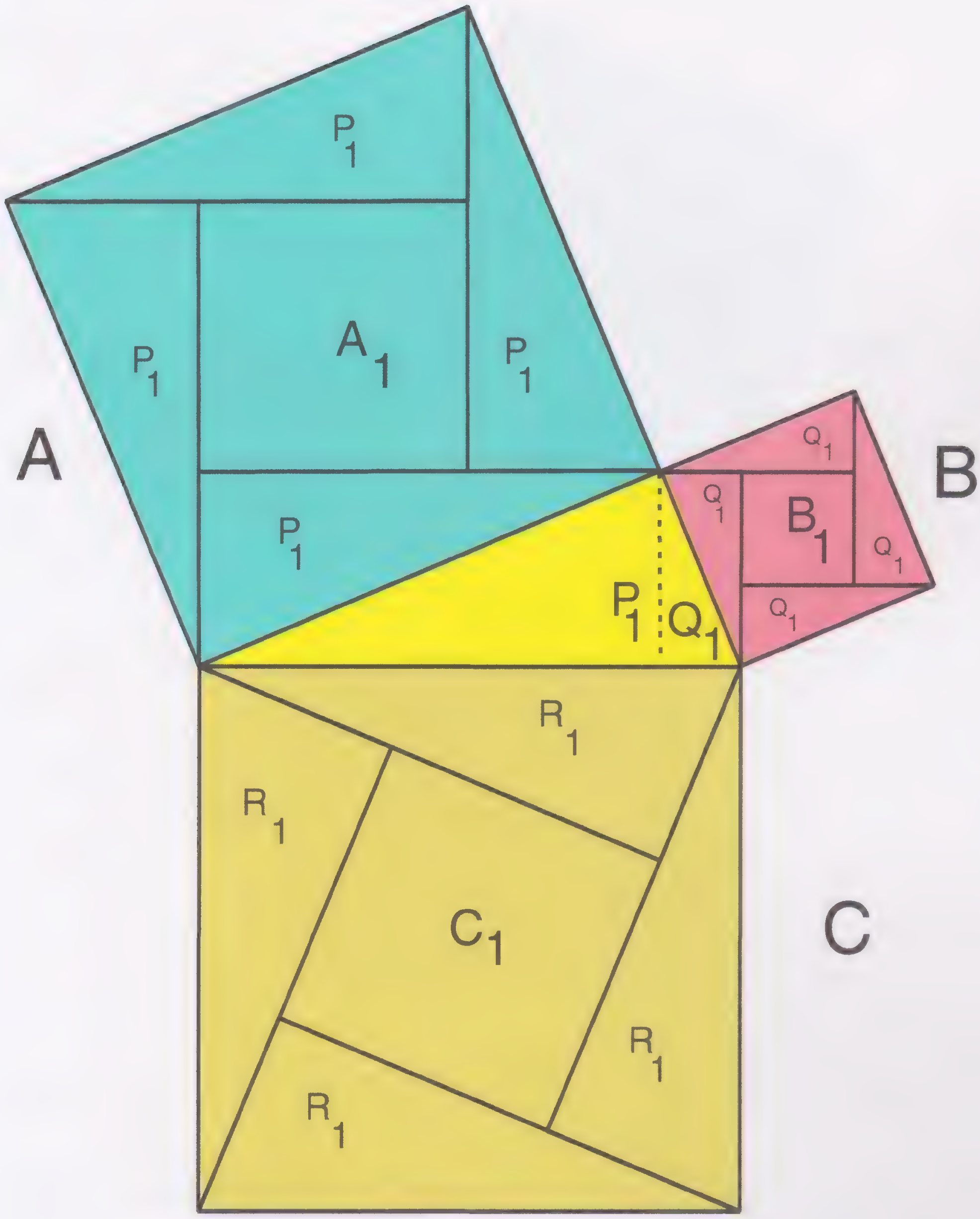
As the sum of the areas P_1 and Q_1 is equal to R_1, we see that the difference between C and (A+B) is equal to the difference between C_1 and (A_1+B_1): from both we have subtracted the same, that is, four times R_1, or four times (P_1+Q_1).

In other words:

$$C - (A+B) = C_1 - (A_1+B_1).$$

What may be said now about C_1, A_1 and B_1?

The triangle with its sides congruent to the sides of the squares with areas A_1, B_1 and C_1 is right-angled, as Figure 7.6 implies.



b

Figure 7.5

When we now apply the same idea to the smaller elephants' defence designs (Figure 7.6c), we find:

$$C_1 - (A_1+B_1) = C_2 - (A_2+B_2).$$

This step may be repeated infinitely:

$$C - (A+B) = C_1 - (A_1+B_1) = C_2 - (A_2+B_2) = \ldots = C_{100} - (A_{100}+B_{100}) = \ldots .$$

A_n, B_n and C_n are becoming smaller and smaller for higher values of n. They converge to 0.

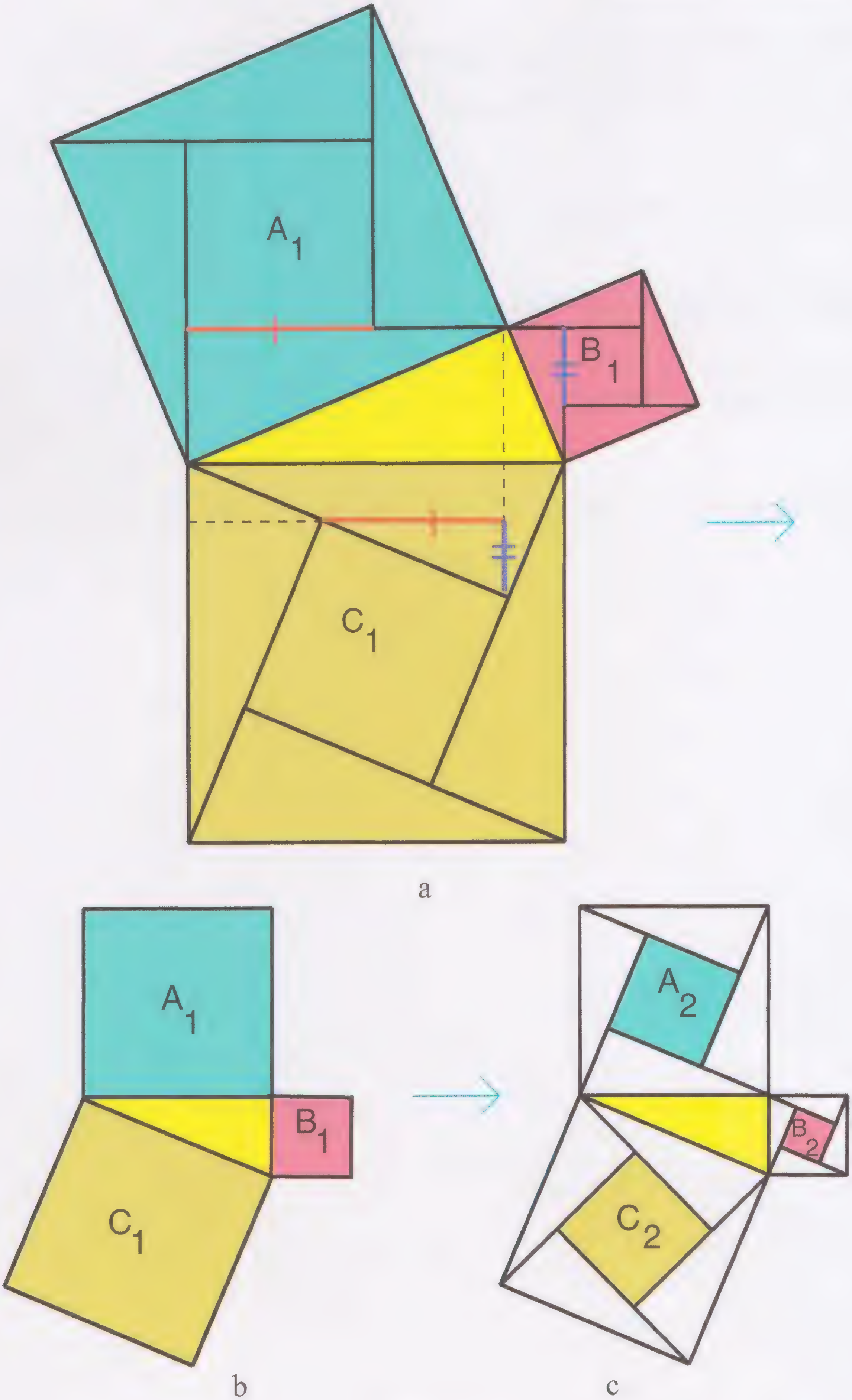


Figure 7.6

By consequence, $C_n - (A_n+B_n)$ converges also to 0.

And in this way, $C - (A+B)$ may only be equal to 0.

In other words, we have shown that

$$A+B = C,$$

that is, the Theorem of Pythagoras.

We may say also that

$$A = 4\sum P_i, B = 4\sum Q_i \text{ e } C = 4\sum R_i$$

and, as $P_i + Q_i$ is equal to R_i, we conclude immediately that:

$$A+B = C.$$

Chapter 8

A NEW PROOF RELATED TO AN ANCIENT EGYPTIAN DECORATION TECHNIQUE *

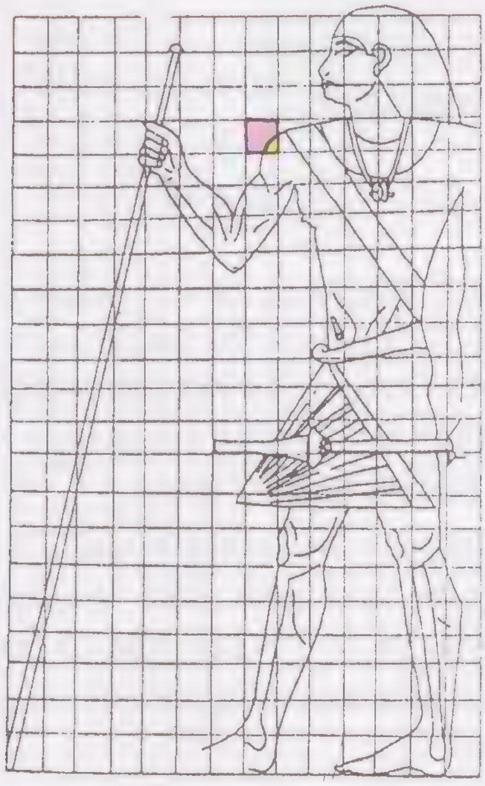


Figure 8.1

From at least the Middle Kingdom (2040-1782 B.C.) onwards, Egyptian artists covered the walls of temples and tombs to be decorated, with a squared grid of red lines used to obtain the correct proportions of figures when transferring small preliminary drawings to the large wall spaces. As the work was completed, the grid lines on the walls were either destroyed or they were covered over with paint.

* The author found this proof in 1986. See (Gerdes, 1986c). Compare with Naber's proof (Van der Waerden, 1983, p. 30) and with the 95th proof included in Loomis' book (1972, p. 84).

Figure 8.1 gives an example, where surviving traces of the original grid have been completed to run over the whole design (Robins, 1986, p. 33; Wilson, 1986, p. 13).

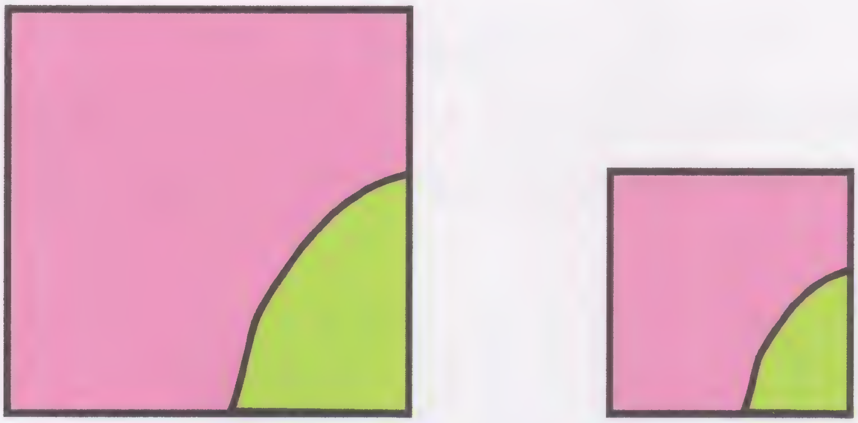
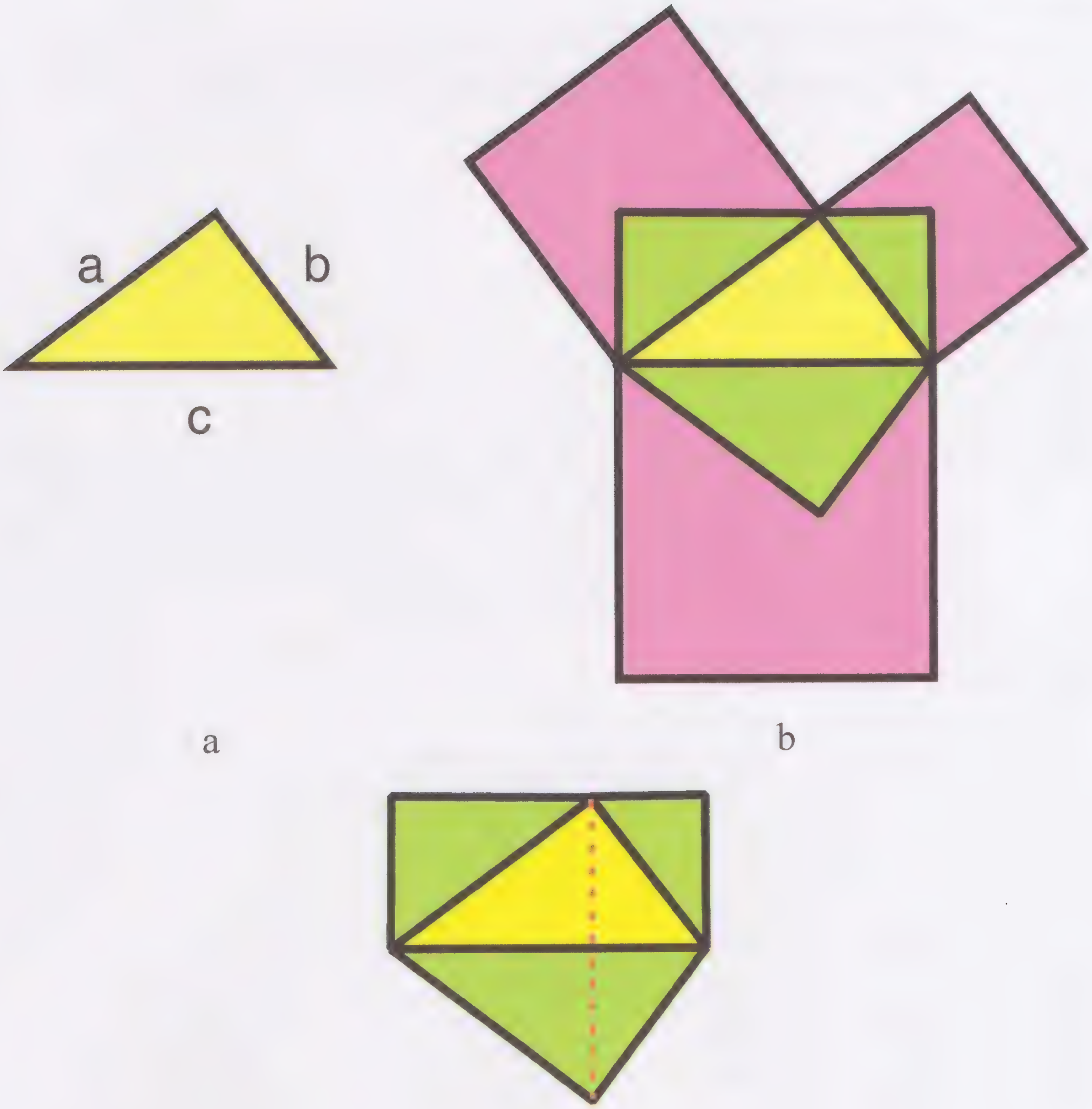


Figure 8.2

Let us look (see Figure 8.2) at one of the squares (8^{th} from the left, 4^{th} from the top in Figure 8.1). It is the enlarged version of a square on the original draft. The part occupied by the shoulder on each square is the same: if it covers **x%** on the small original square, it also covers **x%** on the large wall square.

Now consider arbitrary similar right-angled triangles and squares on their hypotenuses, as in Figure 8.3. As before the right-angled triangles occupy the same proportion (***p***) of the respective squares. Their areas are respectively pa^2, pb^2 and pc^2, where ***a***, ***b***, and ***c*** denote the sides of the squares.

Figure 8.3



a b

c

Figure 8.4

Consider now an arbitrary right-angled triangle with sides *a*, *b* and *c* (see Figure 8.4a) and construct on its sides, as in Figure 8.4b, squares and triangles similar to the given right-angled triangle. As the area of the larger triangle is equal to the sum of the areas of the other triangles constructed in this way (see Figure 8.4c), we find

$$pc^2 = pa^2 + pb^2 ,$$

and, therefore:

$$c^2 = a^2 + b^2 .$$

In other words, we have found another proof for the Pythagorean proposition.

SOURCES OF INFORMATION FOR THE ELABORATION OF THE DRAWINGS

Fig.1.1: Wilson, p. 92, 93;
Fig.1.2: Fortová-Sámalová, T. XXIII;
Fig.1.3: Petrie, p. 31;
Fig.1.4: Fortová-Sámalová, T. XXII
Fig.1.11: Fortová-Sámalová, T. XXIII
Fig.3.1: a, b, c, : Wilson, p. 8, 9, 85; d, e, g, h, i, j, k, l, m, n: Williams, p. 28, 6, 28, 15, 16, 49, 24, 29, 17, 65; f: Bastin, p. 147; j: Cole & Aniakor, p. 46; o, p, r, s, t: Mveng, p. 51, 34, 52, 79, 25; q: Denyer, p. 121; u: Baumann, p. 61
Fig.3.2: Martins, p. 87
Fig.3.14: Redinha, p. 74;
Fig.3.15: Fontinha, p. 207, 239, 227
Fig.3.17: Fontinha, p. 175, 83, 243, 271, 273
Fig.3.22: Meurant, p. 199
Fig.3.27: Mveng, p. 79
Fig.4.1: Meurant, p. 179, 171
Fig.4.2: Meurant, p. 168;
Fig.4.4: Meurant, p. 114, 126, 144, 176
Fig.4.5: Meurant, p. 176;
Fig.4.6: Meurant, p. 176
Fig.5.2: Wilson, p. 23
Fig.5.3: Basket in the collection of the author
Fig.5.4a: Bracelet in the collection of the author
Fig.5.4b: Hauenstein, p. 56
Fig.6.3a: Bastin, p. 116
Fig.6.32b: Basket in the Ethnographical Museum of Budapest (Hungary)
Fig.7.1: Mveng, p. 23, 106, 54, 39, 57, 57, 126, 98, 100, 72, 31, 126
Fig.8.1: Robins, p. 33; Wilson, p. 13

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Basket (Kenya)
(Author's collection)

Basket (Swaziland)
(Author's collection)

Basket (Botswana)
(Author's collection)

Afterword: Review by Jens Hoyrup *

(Section for Philosophy and Science Studies,
Roskilde University, Denmark)

In the author's words, the present book shows "how diverse African ornaments and artefacts may be used to create an attractive context for the discovery and the demonstration of the Pythagorean theorem and of related ideas and propositions." The objective is not to trace the kinds of mathematical knowledge possessed by the producers of these ornaments and artefacts but to provide a framework that may help "to surpass the cultural-psychological learning blockage" that cause "African countries [to be] faced with relatively low 'levels of attainment' in mathematics" by incorporating elements of "ethnomathematics – all types of mathematical activities and reasoning found in daily life – into the curriculum"; these elements are also meant to be "used as ***starting points*** for playing and doing interesting mathematics in and around the classroom." (All quotations from p. 3 – author's emphasis; a few geometric patterns from ancient Egypt occur, but the choice of ornaments and artefacts of recent date shows that sub-Saharan Africa is meant). As far as can be assessed by somebody who never taught in a sub-Saharan classroom this purpose is eminently well served by the book. The reviewer was astonished by the number of different heuristic proofs that could be derived from or impressed on the material. Most of these are evidently of the cut-and-paste type, but others involve the proportionality of similar figures and even considerations of limits. Teachers and textbook authors will thus find substance and an abundance of ideas for the introduction of many essential aspects of geometrical reasoning, not least geometrical reasoning about real-world phenomena. There is no reason that only African teachers should draw on this inspiration, and that fear of exotism should lead didacticians from other parts of the world to

* Review published in *Zentralblatt für Mathematik*, 1996 (Zbl 0840.01001).

dismiss it. In general, the so-called geometric art of non-Modern cultures can be categorized into two types: One is "impressionistic," its aim (as far as geometry is concerned) is the global visual impression; the Greek vase painting of the Geometric Period is an exquisite example; in earlier specimens from the Greek and Aegean region one may find, e.g., a mixture of local 6- and 7-fold symmetries, or chessboard-patterns where a few cases get the "wrong" colour. The other type bears witness of deliberate explorations of symmetries and other formalizeable properties of figures; its actual drawings need not be very precise, but they contain an underlying formal structure. The "impressionistic'' type, whether visually exquisite or crude, is evidently unfit as a basis for mathematical reasoning. But all the examples explored by the author (and sub-Saharan geometrical decoration in broad average as far as the reviewer is aware) belong to the second type. Together with certain local traditions (e.g., Peruvian Indians), sub-Saharan African geometrical art thus possesses a universal value of which the mathematics education of the global village should take advantage.

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* *African Pythagoras: A study in Culture and Mathematics Education*, Mozambican Ethnomathematics Research Centre (MERC), Maputo & Lulu, Morrisville NC, 2011, 124 pp.
  [First edition: Universidade Pedagógica, Maputo, 1994, 103 pp.]
* *Tinhlèlò, Interweaving Art and Mathematics: Colourful Circular Basket Trays from the South of Mozambique*, MERC, Maputo & Lulu, Morrisville NC, 2010, 132 pp.
  (Foreword: Aires Aly, Prime Minister of Mozambique)
* *Otthava: Making Baskets and Doing Geometry in the Makhuwa Culture in the Northeast of Mozambique*, MERC, Maputo & Lulu, Morrisville NC, 2010, 290 pp. & *Otthava Images in Colour: A Supplement*, 68 pp.
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  (Preface: Alcido Guenha, Minister of Education of Mozambique; Foreword: Emília Nhalivilo, Universidade Pedagógica, Mozambique; Afterword: Dr. Hippolyte Foffack, Nelson Mandela Institution & Worldbank; Afterthoughts: Dr. Joan Conolly, Durban University of Technology, South Africa)

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Paulus Gerdes has been a professor of mathematics and ethnomathematics at the Eduardo Mondlane University, and the 'Universidade Pedagógica' in Mozambique. He has served as Head of the Department of Mathematics and Physics (1981-1983), as Dean of the Faculty of Education (1983-1987), and as Dean of the Faculty of Mathematics (1987-1989) of the Eduardo Mondlane University. He was President (Rector) of the 'Universidade Pedagógica' (1989-1996). He was a visiting professor at the University of Georgia from 1996 to 1998 (Athens, USA). Since 1998 he is director of the Mozambican Ethnomathematics Research Centre – Mathematics, Culture and Education in Maputo. From 2000 to 2004 he was also senior advisor to the Minister of Education. In 2006-2007 he was president of the founding commission of the Lúrio University, Mozambique's third public university established in Nampula.

Among his international functions may be mentioned that Dr. Paulus Gerdes has been chair of the international Commission for the History of Mathematics in Africa (since 1986) and President of the International Association for Science and Cultural Diversity (2001-2005). In 2000, he succeeded the Brazilian Ubiratan D'Ambrosio as President of the International Study Group for Ethnomathematics. He is a fellow of the International Academy for the History of Science, and was elected, in 2005, Vice-President of the African Academy of Sciences, responsible for the Southern African region (re-elected in 2011).

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The aim of “AFRICAN PYTHAGORAS: A study in culture and mathematics education” is to show how diverse African ornaments and artefacts may be used to create an attractive context for the discovery and the demonstration of the Pythagorean Theorem and of related ideas and propositions.

“Teachers and textbook authors will … find substance and an abundance of ideas for the introduction of many essential aspects of geometrical reasoning, not least geometrical reasoning about real-world phenomena. There is no reason that only African teachers should draw on this inspiration, … [as] sub-Saharan African geometrical art … possesses a universal value of which the mathematics education of the global village should take advantage.”

Jens Hoyrup, Roskilde University, Denmark

Keywords:

Africa, African culture, African art, Pythagoras, Pythagorean Theorem, Pappus, geometry, proof, heuristics, magic square, Latin square, trigonometry, fractal, mathematics, mathematics education, teacher education, ethnomathematics, ethnogeometry

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